Tetrahelix Coordinate Calculations

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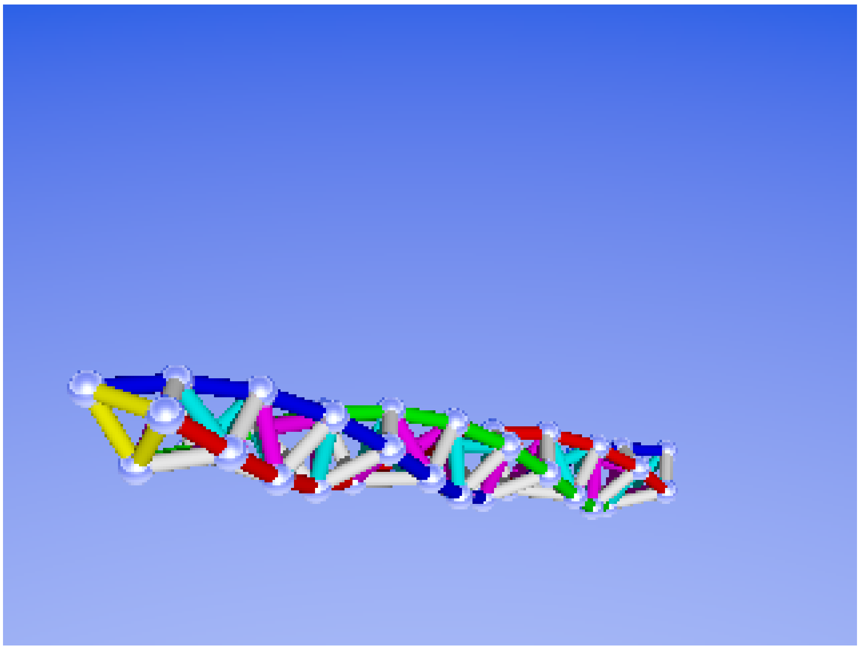
Abstract

This note shows the derivation of the coordinates of the tetrahedra in a tetrahelix. This note expands upon content at rwgrayprojects.com.

Spiral Curve

I was first introduced to tetrahedral spirals when playing with geomag toys, which consists of small rod magnets and ball bearings. I called these tetrahedral trusses initially, but have since found these structures described as tetrahelixes (Buckminister Fuller) and Bernal Spirals. My webpages describing the different iterated tetrahedral structures are at http://www.kurtnalty.com/Tetrahedral/



My initial approach to building tetrahelices began with a triangle, then added one vertice at a time using cross products and vector addition. This results in models propagating in odd angles from the 3D rendering coordinate system. RW Gray correctly points out that since these helices are a spiral, we might as well align the spiral axis with the z axis, for example, and find the cylindrical coordinates for the vertices.

Align the axis of the spiral with z. Use a unit radius for the bounding cylinder. Use a step height h. Use a step angle θ. Call the tetrahedron edge length l. Starting with the first vertex on the unit x coordinate, I have coordinates

$$\begin{aligned} V0: &\quad (1,0,0) \\ V1: &\quad (\cos(\theta), \sin(\theta), h) \\ V2: &\quad (\cos(2\theta), \sin(2\theta), 2h) \\ V3: &\quad (\cos(3\theta), \sin(3\theta), 3h) \end{aligned}$$

The squared distances between these vertices will all be same at l^2.

$$\begin{aligned} V1 - V0: &\quad (\cos(\theta) - 1)^2 + \sin^2(\theta) + h^2 = l^2 \\ V2 - V0: &\quad (\cos(2\theta) - 1)^2 + \sin^2(2\theta) + (2h)^2 = l^2 \\ V3 - V0: &\quad (\cos(3\theta) - 1)^2 + \sin^2(3\theta) + (3h)^2 = l^2 \end{aligned}$$

Simplifying these expressions yields

$$\begin{aligned} V1 - V0: &\quad 2 - 2\cos(\theta) + h^2 = l^2 \\ V2 - V0: &\quad 2 - 2\cos(2\theta) + 4h^2 = l^2 \\ V3 - V0: &\quad 2 - 2\cos(3\theta) + 9h^2 = l^2 \end{aligned}$$

Eliminate l^2

$$\begin{aligned} 2 - 2\cos(2\theta) + 4h^2 &= 2 - 2\cos(\theta) + h^2 \\ 2 - 2\cos(3\theta) + 9h^2 &= 2 - 2\cos(\theta) + h^2 \end{aligned}$$

Simplify and isolate h^2

$$\begin{aligned} 3h^2 &= -2\cos(\theta) + 2\cos(2\theta) \\ 8h^2 &= -2\cos(\theta) + 2\cos(3\theta) \\ 4h^2 &= -\cos(\theta) + \cos(3\theta) \\ 12h^2 &= -8\cos(\theta) + 8\cos(2\theta) = -3\cos(\theta) + 3\cos(3\theta) \end{aligned}$$

Collect terms and use multiple angle formulas to get to simple $\cos(\theta)$.

$$\begin{aligned} 3\cos(3\theta) - 8\cos(2\theta) + 5\cos(\theta) &= 0 \\ 3(4\cos^3(\theta) - 3\cos(\theta)) - 8(2\cos^2(\theta) - 1) + 5\cos(\theta) &= 0 \\ 12\cos^3(\theta) - 16\cos^2(\theta) - 4\cos(\theta) + 8 &= 0 \\ (\cos(\theta) - 1) * (\cos(\theta) - 1) * (12\cos(\theta) + 8) &= 0 \end{aligned}$$

We have two trivial solutions ($\cos(\theta) = 1, \theta = 0$), and the desired solution $\cos(\theta) = -2/3$.

We now find h^2.

$$\begin{aligned} h^2 &= \frac{2}{3}(\cos(2\theta) - \cos(\theta)) \\ &= \frac{2}{3}\left(2\cos^2(\theta) - 1 - \cos(\theta)\right) \\ &= \frac{2}{3}\left(2 * \frac{4}{9} - 1 + \frac{2}{3}\right) \\ &= \frac{10}{27} \end{aligned}$$

Solve for l^2

$$
\begin{aligned}
l^2 &= 2 - 2\cos(\theta) + h^2 \\
&= \left(2 + 2 * \frac{2}{3} + \frac{10}{27}\right) \\
&= \frac{100}{27}
\end{aligned}
$$

Summary, for unit radius cylinder:

$$
\begin{aligned}
\theta &= \cos^{-1}\left(-\frac{2}{3}\right) \\
r &= 1 \\
h &= \sqrt{\frac{10}{27}} \\
l &= \sqrt{\frac{100}{27}} = h\sqrt{10}
\end{aligned}
$$

Alternatively, for unit tetrahedral length:

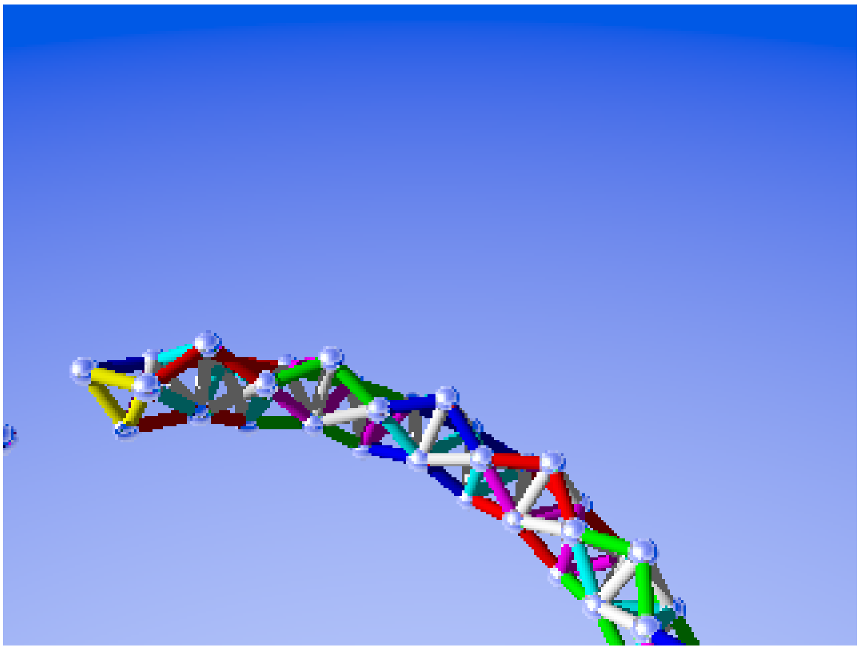
$$
\begin{aligned}
\theta &= \cos^{-1}\left(-\frac{2}{3}\right) \\
r &= \frac{3\sqrt{3}}{10} \\
h &= \frac{1}{\sqrt{10}} \\
l &= 1
\end{aligned}
$$

Observations

As noted by RW Gray, the angular step per vertice is an irrational number. Consequently, these do not form a perfectly periodic array. [This apparently affects the appearance of x-ray diffraction images.]

TetraRings

Alternating sections of positive helicity and negative helicity gives a construction with a zero net torsion. Using geomags, this structure closes after 32 sections. The question arises whether this is due to closure or manufacturing and assembly tolerances.



Examining the ring, the outer ring has the same spacing as two adjacent spine elements in the tetrahelix. Consequently, I can quickly write the coordinates for these three points, then determine the radius of the circumcircle for these points, then obtain the angular separation for these points, and in this fashion, determine the angle spanned by two links of the outer perimeter of the tetraring.

I will use vertices $V(-3), V(0), V(+3)$ for simplicity.

The coordinates of these elements for a unit edge spiral around the Z axis are

$$\begin{aligned}
V(n) &: (r\cos(n\theta), r\sin(n\theta), nh) \\
V(-3) &: \left(\frac{2.2}{3\sqrt{3}}, \frac{-0.7}{3}\sqrt{\frac{5}{3}}, \frac{-3}{\sqrt{10}}\right) \\
V(0) &: \left(\frac{3\sqrt{3}}{10}, 0, 0\right) \\
V(3) &: \left(\frac{2.2}{3\sqrt{3}}, \frac{0.7}{3}\sqrt{\frac{5}{3}}, \frac{3}{\sqrt{10}}\right)
\end{aligned}$$

where $r = 0.3\sqrt{3}$, $\theta = \cos^{-1}(-2/3)$, and $h = 1/\sqrt{10}$. I've used

$$\begin{aligned}
\cos\theta &= -2/3 \\
\sin\theta &= \sqrt{5/9} \\
\cos 3\theta &= 22/27 \\
\sin 3\theta &= \frac{7\sqrt{5}}{27}
\end{aligned}$$

This triangle has sides $(a, b, c) = (1, 1, \sqrt{107/27})$. The area of this triangle is $K = \sqrt{107}/108$. The radius of the circumscribed circle passing through these three points is

$$
\begin{aligned}
R &= \frac{abc}{K} \\
&= 3\sqrt{3}
\end{aligned}
$$

The angle subtended over one outer segment is

$$
\begin{aligned}
\theta &= \sin^{-1}\left(\frac{\frac{1}{2}\sqrt{\frac{107}{27}}}{\sqrt{27}}\right) \\
&= \sin^{-1}\left(\frac{\sqrt{107}}{54}\right) \\
&= 0.192748
\end{aligned}
$$

This is about 1.8% shy of $2\pi/32 = 0.19634954$. Just as with the tetrahelix, we don't have a perfectly periodic structure, although we are close enough for manufacturing.