Complex, Quaternion and Octonion Formulas Using Summation Notation

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Magnitude of Vectors and Products

The squared magnitude of an arbitrary vector a_i, using summation notation is

$$|a|^2 = a_i a^i = a_i a^j \delta^i_j$$

The product of two arbitrary vectors, yielding another vector, is

$$c_k = a_i b_j P^{ij}_k$$

The division algebras of complex numbers, quaternions and octonions are often defined using a truth table approach where the index of the product basis is the exclusive or of the indices of the two factors, times a sign value. For example, mapping (1,i,j,k) to (0,1,2,3), we find the elements of the product i*j = +k selecting the k from $(1 \wedge 2) = 3$, while the sign is specified separately. (The $\wedge$ is the bit-by bit exclusive or operator from C.)

Consequently, for these division algebras, the product tensor P^{ij}_k can be factored as

$$P^{ij}_k = S^{ij} \delta^{i \wedge j}_k$$

A characteristic of the division algebras is normality of the products, where the magnitude squared of the product is equal to the product of the input magnitudes squared. $c = ab$ implies $|c|^2 = |a|^2 |b|^2$.

Expressing this using summation notation, we have

$$\begin{aligned} c_k c^k &= \left(a_i b_j S^{ij} \delta_k^{i\wedge j}\right)\left(a^m b^n S_{mn} \delta^k_{m\wedge n}\right) = a^i a_i b^j b_j \\ &= a_i a^m b_j b^n S^{ij} S_{mn} \delta_k^{i\wedge j} \delta^k_{m\wedge n} = a_i a^m b_j b^n \delta^i_m \delta^j_n \end{aligned}$$

The right hand two terms can be subtracted, and the left delta contracted, to yield

$$a_i a^m b_j b^n S^{ij} S_{mn} \delta^{i\wedge j}_{m\wedge n} - a_i a^m b_j b^n \delta^i_m \delta^j_n = 0$$
$$a_i a^m b_j b^n \left(S^{ij} S_{mn} \delta^{i\wedge j}_{m\wedge n} - \delta^i_m \delta^j_n\right) = 0$$

Perfect Square Terms

For the squared terms, we have $i = m$ and $j = n$. The term in parenthesis above disappears, and

$$S^{ij} S_{mn} \delta^{i\wedge j}_{m\wedge n} - \delta^i_m \delta^j_n = S^{ij} S_{ij} - 1 = 0$$

We thus know that all sign array elements are +1 or -1, and not complex or zero.

Rectangular Sums and Symmetry

For the non-square terms, where $i \neq m$ or $k \neq n$, we have four terms of interest that will sum to zero. We are here looking at individual components with no sum implied.

$$\begin{aligned} a_i a^m b_j b^n S^{ij} S_{mn} \delta^{i\wedge j}_{m\wedge n} + a_i a^m b_n b^j S^{in} S_{mj} \delta^{i\wedge n}_{m\wedge j} + \\ a_m a^i b_j b^n S^{mj} S_{in} \delta^{m\wedge j}_{i\wedge n} + a_m a^i b_n b^j S^{mn} S_{ij} \delta^{m\wedge n}_{i\wedge j} &= 0 \end{aligned}$$

Being a bit sloppy with summation notation, given that these items are not really covariant or contravariant as written,

$$a^i a^m b^j b^n \left(S_{ij} S_{mn} \delta^{i\wedge j}_{m\wedge n} + S_{in} S_{mj} \delta^{i\wedge n}_{m\wedge j} + S_{mj} S_{in} \delta^{m\wedge j}_{i\wedge n} + S_{mn} S_{ij} \delta^{m\wedge n}_{i\wedge j}\right) = 0$$

The $\delta^{i\wedge n}_{m\wedge j}$ imply $i \wedge j \wedge m \wedge n = 0$ Consequently, these are a common factor.

$$a^i a^m b^j b^n \delta^{i\wedge j}_{m\wedge n} \left(S_{ij} S_{mn} + S_{in} S_{mj} + S_{mj} S_{in} + S_{mn} S_{ij}\right) = 0$$

We see that we have two copies of each term in the parenthesis. We get the simple result (with $n = i \wedge j \wedge m$)

$$S_{ij}S_{mn} + S_{in}S_{mj} = 0$$

Conclusion

For the product of magnitudes to equal the magnitude of products, with the simple truth table format for basis multiplication, we have a requirement that the sign matrix elements are either +1 or -1, and we have the requirement that (with $n = i \wedge j \wedge m$)

$$\begin{aligned} S_{ij}S_{mn} + S_{in}S_{mj} &= 0 \\ S_{ij} = -S_{in}S_{mj}/S_{mn} \end{aligned}$$

Given the restricted values of S_{ij}, we can replace multiplication with division, and have the simple formula

$$S_{ij} = -S_{in}S_{mj}S_{mn}$$