Angular Velocity and Momentum

Kurt Nalty

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Introduction

The presentation of angular momentum in the classical physics world is dominated by rigid body mechanical concepts. This, in turn, has led to some poor definitions presented for angular velocity and momentum.

1 Angular Velocity

When dealing with angular velocity, we must first be aware that there are two different points of view involved. One point of view is the angular velocity of a particle as seen by an observer (observed angular velocity). The other point of view is that of the particle itself as it travels (intrinsic angular velocity).

1.1 Observed Angular Velocity

Observed angular velocity depends upon the relative placement of the observer and particle. Different observers will see different angular velocity profiles.

Here we will determine the formula for $d\vec{\theta}$ from geometry, then obtain $\vec{\omega} = d\vec{\theta}/dt$.

Construct a narrow triangle from $\vec{r}$, $d\vec{r}$ and $\vec{r} + d\vec{r}$. Label the narrow angle between $\vec{r}$ and $\vec{r} + d\vec{r}$ as $d\vec{\theta}$. The area of this infinitesimal triangle is

$$\vec{A} = \frac{1}{2}\vec{r} \times (\vec{r} + d\vec{r}) = \frac{1}{2}\vec{r} \times d\vec{r} = \frac{1}{2} r^2 d\vec{\theta} \tag{1}$$

This leads to the definitions

$$d\vec{\theta} = \frac{\vec{r} \times d\vec{r}}{r^2} \tag{2}$$

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} = \frac{\vec{r} \times \vec{v}}{r^2} \tag{3}$$

This $\vec{\omega}$ is the angular velocity of a particle, as seen by an observer at the origin. Different observers at different locations will see different angular velocities.

1.2 Intrinsic Angular Velocity

I start by returning to the Frenet formulas. Curvature is the inverse of the radius of the circle, whose curvature matches the curvature of the trajectory at that tangent. When people are referring to a radius when discussing the angular velocity of a particle, it is really the radius of curvature ($\vec{r} = \vec{n}/\kappa$) to which they are referring.

Given that the tangent (along $\vec{v}$), normal (along $\vec{r}$) and binormal (along $\vec{\omega}$) form an orthogonal triad, we can rewrite the particle velocity and angular velocity equation as

$$\vec{v} = \vec{\omega} \times \vec{r} \rightarrow \vec{\omega} = \frac{1}{r}\vec{n} \times \vec{v} = \kappa\vec{n} \times \vec{v} = \frac{d\vec{u}}{ds} \times \vec{v} \tag{4}$$

There are a few interesting forms for this formula.

$$\begin{aligned} \vec{\omega} &= \frac{d\vec{u}}{ds} \times \vec{v} && (5) \\ \frac{d\vec{\theta}}{dt} &= \frac{d\vec{u}}{ds} \times \frac{d\vec{r}}{dt} && (6) \\ d\vec{\theta} &= \frac{d\vec{u}}{ds} \times d\vec{r} && (7) \\ \frac{d\vec{\theta}}{ds} &= \frac{d\vec{u}}{ds} \times \frac{d\vec{r}}{ds} && (8) \\ \frac{d\vec{\theta}}{ds} &= \frac{d\vec{u}}{ds} \times \vec{u} && (9) \end{aligned}$$

Let's write some expressions for $\vec{\omega}$.

$$\begin{aligned} \vec{\omega} &= \frac{d\vec{u}}{ds} \times \vec{v} && (10) \\ &= \kappa\vec{n} \times \vec{v} && (11) \\ &= \left[\frac{\vec{v} \times (\vec{a} \times \vec{v})}{v^4}\right] \times \vec{v} && (12) \\ &= \left[\frac{\vec{a}v^2 - (\vec{v} \cdot \vec{a})\vec{v}}{v^4}\right] \times \vec{v} && (13) \\ &= \frac{\vec{a} \times \vec{v}}{v^2} && (14) \end{aligned}$$

Now let's check for $\vec{\omega} \times \vec{r} = \vec{v}$, where $\vec{r}$ is the radius of curvature.

$$\begin{aligned} \vec{\omega} \times \vec{r} &= \left(\frac{\vec{a} \times \vec{v}}{v^2}\right) \times \frac{v^2\,(\vec{v} \times (\vec{a} \times \vec{v}))}{|\vec{a} \times \vec{v}|^2} && (15) \\ &= (\vec{a} \times \vec{v}) \times \frac{(\vec{v} \times (\vec{a} \times \vec{v}))}{|\vec{a} \times \vec{v}|^2} && (16) \\ &= \frac{\vec{v}\,(\vec{a} \times \vec{v}) \cdot (\vec{a} \times \vec{v}) - (\vec{a} \times \vec{v})\,(\vec{v} \cdot (\vec{a} \times \vec{v}))}{|\vec{a} \times \vec{v}|^2} && (17) \\ &= \frac{\vec{v}\,(\vec{a} \times \vec{v}) \cdot (\vec{a} \times \vec{v})}{|\vec{a} \times \vec{v}|^2} && (18) \\ &= \vec{v} && (19) \end{aligned}$$

Summarizing, we have two different angular velocities involved in the discussion of trajectories.

One is the angular velocity of a particle on a trajectory as seen by an observer. In three dimensions, this is given by $\vec{\omega} = \vec{r} \times \vec{v}/r^2$.

The other is the intrinsic angular velocity of the particle as experienced by itself, due to the curvature of its trajectory. In three dimensions, this is given by $\vec{\omega} = \vec{a} \times \vec{v}/v^2$.

1.3 Quaternion Angular Velocity in Fourspace

In four dimensional space, rotation is more complex, as we have more planes capable of supporting simultaneous rotation. Quaternion multiplication by a unit vector results in coupled rotations, one about the space vector axis (axis of rotation parallel with space, normal to plane), the other in the plane defined by time and the space vector (axis perpendicular to space vector and time).

Begin by looking at the expressions for three dimensional angular velocity with respect to path length.

$$\frac{d\vec{\theta}}{ds} = \frac{d\vec{u}}{ds} \times \vec{u} = \kappa\vec{n} \times \vec{u} \tag{20}$$

We see that the magnitude of $d\theta/ds$ is κ, while the direction is given by $\vec{n} \times \vec{u}$.

Extending the same process to quaternion space, we have

$$\begin{aligned}
\frac{d\tilde{\theta}}{ds} &= \frac{d\tilde{u}}{ds}\tilde{u} && (21)\\
&= \kappa\tilde{n}\tilde{u} && (22)\\
&= \kappa\tilde{u}\vec{g}\tilde{u} && (23)\\
&= \kappa\tilde{u}\left(\frac{\vec{a} + \vec{a} \times \vec{v}}{|\vec{a} + \vec{a} \times \vec{v}|}\right)\tilde{u} && (24)\\
&= \frac{|\vec{a} + \vec{a} \times \vec{v}|}{(1+v^2)^{3/2}}\tilde{u}\left(\frac{\vec{a} + \vec{a} \times \vec{v}}{|\vec{a} + \vec{a} \times \vec{v}|}\right)\tilde{u} && (25)\\
&= \tilde{u}\left[\frac{\vec{a} + \vec{a} \times \vec{v}}{(1+v^2)^{3/2}}\right]\tilde{u} && (26)
\end{aligned}$$

While not yet terribly insightful, the expression for the angular velocity per unit distance in quaternion space is

$$\frac{d\tilde{\theta}}{ds} = \frac{-2(\vec{v} \cdot \vec{a})(1 + \vec{v}) + (\vec{a} + \vec{a} \times \vec{v})\left(1 + v^2\right)}{(1+v^2)^{5/2}} \tag{27}$$

The angular velocity per unit time is

$$\begin{aligned}
\frac{d\tilde{\theta}}{dt} &= \frac{-2(\vec{v} \cdot \vec{a})(1 + \vec{v}) + (\vec{a} + \vec{a} \times \vec{v})\left(1 + v^2\right)}{(1+v^2)^2} && (28)\\
\frac{d\tilde{\theta}}{dt} &= \frac{-2(\vec{v} \cdot \vec{a})(1 + \vec{v})}{(1+v^2)^2} + \frac{(\vec{a} + \vec{a} \times \vec{v})}{(1+v^2)} && (29)
\end{aligned}$$

The magnitude of the angular velocity per unit time is

$$|\tilde{\omega}| = \frac{|\vec{a} + \vec{a} \times \vec{v}|}{1 + v^2} \tag{30}$$

Finally, note that angular velocity has the same units as energy and momentum.

2 Angular Momentum

Classically, the angular momentum of a body is defined by

$$\vec{L} = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p} \tag{31}$$

where $\vec{L}$ is the angular momentum, m is the mass, $\vec{v}$ is the velocity, and $\vec{r}$ is the vector from the origin (usually at the axis of rotation) to the particle. The term $\vec{p}$ represents the linear momentum of the particle.

I don't want to get bogged down by speculating about mass yet, so I'll leave m alone for the moment. Likewise, from the previous discussion of intrinsic angular velocity, I identified r as the radius of curvature $r = 1/\kappa$.

The time rate of change of angular momentum is torque.

$$\vec{T} = \frac{d\vec{L}}{dt} = \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{f} \tag{32}$$

where $\vec{f}$ is the force on the particle.

More interesting, the curl of classical angular momentum is proportional to linear momentum.

$$\vec{\nabla} \times \vec{L} = -2m\vec{v} = -2\vec{p} \tag{33}$$

When two equal opposing forces separated by a distance are applied to an object, we have a torque due to this couple.

$$\vec{T} = (\vec{r}_1 - \vec{r}_2) \times \vec{f} \tag{34}$$

If we substitute the radius of curvature in the definition of $\vec{L}$, we obtain the rather strange looking equation

$$\vec{r} = \frac{1}{\kappa}\vec{n} = \frac{v^2\,(\vec{v} \times (\vec{a} \times \vec{v}))}{|\vec{a} \times \vec{v}|^2} \tag{35}$$

$$\vec{L} = \frac{mv^4(\vec{a} \times \vec{v})}{|\vec{a} \times \vec{v}|^2} \tag{36}$$

This can be recognized as

$$\vec{L} = m\frac{1}{\kappa^2}\vec{\omega} \tag{37}$$

$$= m\vec{r} \times (\vec{r} \times \vec{\omega}) \tag{38}$$

$$= m\vec{r} \times \vec{v} \tag{39}$$

3 Action and the Quaternion Derivative

Action is lightly addressed in most introductory texts. When used, action is presented as part of an integral of energy over time in the context of deriving the Hamilton/Lagrange equations. I believe that action is more fundamental than currently credited. I also believe that some confusion is possible over whether energy is a partial or whole derivative of action versus time. I further believe that action is part of a four-vector, whose spatial components are fundamental to angular velocity and momentum.

Some interesting possibilities to consider are

$$E = \frac{da}{dt} \quad ? \tag{40}$$

$$E = \frac{\partial a}{\partial t} \quad ? \tag{41}$$

$$(a, \vec{L}) \rightarrow (E, \vec{p}) \rightarrow (P, \vec{F}) \quad ? \tag{42}$$

4 Quantum L Operator Miscellaneous Note

The operator $\vec{r} \times \vec{\nabla}$ when applied to itself remains a first order operator.

$$\left(\vec{r} \times \vec{\nabla}\right) \times \left(\vec{r} \times \vec{\nabla}\right) = -\left(\vec{r} \times \vec{\nabla}\right) \tag{43}$$