Sample Calculations using Weber Force Law and Free Charges

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Abstract

Weber and Gauss (1831 and later) extended Ampere's magnetic force law to include a potential formulation for the force between two charged particles. For macroscopic engineering work, their formula is indistinguishable from the Biot-Savart law or Ampere law. This note examines the Weber law as a candidate for particle to particle interaction.

Ampere's and Weber's Force Laws

Wilhelm Eduard Weber literally set the standard in fundamental electromagnetic experimental work, from measuring terrestrial magnetism to setting standards for resistance, current, magnetic fields and voltage. Weber (and Gauss) were also the first, to my knowledge, to achieve a potential formulation for electromagnetic interactions.

Prior to Weber, explanations for electromagnetic forces presented the appropriate force laws, consistent with measurement, and the matter was settled. From an engineering point of view, this is a very reasonable approach. However, with the concept of the conservation of energy, and the origination of forces as gradients in energy density, Weber wanted to do more. Starting with Ampere's law, he examined different configuration of charge carriers, and using the principle of virtual work, found a potential formulation that leads to Ampere's law.

Using MKS units, Ampere's differential force law between two infinitesimal current elements is

$$\begin{aligned}\frac{d^2\vec{F}}{ds\; ds'} &= \frac{\mu II'}{4\pi r^2}\left(2r\frac{\partial^2 r}{\partial s\;\partial s'} - \frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right)\vec{a}_r \\ &= \frac{\mu II'}{4\pi}\frac{1}{r^2}\left[3\left(\vec{a}_r\cdot\vec{u}\right)\left(\vec{a}_r\cdot\vec{u}'\right) - 2\left(\vec{u}\cdot\vec{u}'\right)\right]\vec{a}_r\end{aligned}$$

where ds and ds' are differential lengths along the two conductors, r is the separation between the current elements, $\vec{u}$ and $\vec{u}'$ are unit tangents to the curves, $\vec{a}_r$ is the unit vector in the direction between the two current elements, and I and I' are the current values.

Dropping from current elements to elementary charges and time derivatives, Weber's force law is

$$\vec{F} = \frac{q_i q_j}{4\pi\epsilon_0}\frac{\vec{r}_{ij}}{r_{ij}^3}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right)$$

where $\epsilon = 8.854 \cdot 10^{-12}$ F/m, r is the radial separation, $\dot{r}$ is the time rate of change of radial separation, $\ddot{r}$ is the second time derivative of the separation, and $c = 2.998 \cdot 10^8$ m/s is the speed of light.

We can re-write this using vector notation. Define relative positions, velocities and accelerations.

$$\begin{aligned}\vec{R} &= \vec{r}_1 - \vec{r}_2 \\ \vec{a}_r &= \frac{\vec{R}}{\sqrt{\vec{R}\cdot\vec{R}}} \\ \vec{V} &= \vec{v}_1 - \vec{v}_2 \\ \vec{A} &= \vec{a}_1 - \vec{a}_2\end{aligned}$$

Calculate time derivatives of radial separation.

$$\begin{aligned}\vec{R}\cdot\vec{R} &= r^2 \\ \frac{d}{dt}\vec{R}\cdot\vec{R} &= \frac{d}{dt}r^2 \\ 2\vec{R}\cdot\frac{d\vec{R}}{dt} = 2\vec{R}\cdot\vec{V} &= 2r\frac{dr}{dt} = 2r\dot{r} \\ \dot{r} &= \vec{a}_r\cdot\vec{V}\end{aligned}$$

Now for the second derivative.

$$\begin{aligned}
\frac{d}{dt}\dot{r} &= \frac{d}{dt}\left(\frac{\vec{R}\cdot\vec{V}}{r}\right) \\
&= \frac{1}{r}\left(\vec{R}\cdot\vec{A}+\vec{V}\cdot\vec{V}\right) - \frac{1}{r^2}\left(\vec{R}\cdot\vec{V}\right)\frac{dr}{dt} \\
\ddot{r} &= \frac{1}{r}\left(\vec{R}\cdot\vec{A}+\vec{V}\cdot\vec{V} - \left(\vec{a}_r\cdot\vec{V}\right)^2\right)
\end{aligned}$$

As an aside, the last two terms in the parenthesis can be spotted as the squared magnitude of $\vec{a}_r \times \vec{V}$.

Substituting these in the Weber force equation, we have (in vector component format)

$$\begin{aligned}
\vec{F} &= \frac{q_i q_j}{4\pi\epsilon_0}\frac{\vec{r}_{ij}}{r_{ij}^3}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right) \\
&= \frac{q_i q_j}{4\pi\epsilon_0}\frac{\vec{r}_{ij}}{r_{ij}^3}\left(1 + \left(\frac{V}{c}\right)^2 - \frac{3}{2}\left(\frac{\vec{a}_r\cdot\vec{V}}{c}\right)^2 + \frac{\vec{R}\cdot\vec{A}}{c^2}\right)
\end{aligned}$$

Weber's potential, from which the above is derived by radial gradient, is

$$U = \frac{q_i q_j}{4\pi\epsilon}\frac{1}{r_{ij}}\left(1 - \frac{\dot{r_{ij}}^2}{2c^2}\right)$$

or in vector notation

$$U = \frac{q_i q_j}{4\pi\epsilon}\frac{1}{r}\left(1 - \frac{(\vec{R}\cdot\vec{V})^2}{2r^2c^2}\right)$$

Historical Loss of Interest in Weber's Force Law

Weber's force law fell into neglect for several reasons.

Weber used Fechner's hypothesis that current was carried in conductors by positive and negative charges both carrying half the current. We know this hypothesis to be false, as the electrons are mobile but the nucleons fixed for metals. With the dismissal of the Fechner hypothesis, many people dismissed the Weber formula as also necessarily invalid. This was unfortunate. Andre Assis, in his book *Weber's Electrodynamics (1994)* demonstrates that the only necessary requirement to achieve Ampere's law from Weber's law is that of neutral conductors.

Weber's force law takes more effort for engineering calculations than Biot-Savart. For engineering calculations, the three Weber interactions (e1-e2,p1-e2,e1-p2) simplify to Ampere's law. Ampere's law, in turn, requires a few more add/multiplies than Biot-Savart. In the engineering regime, when results are identical, the calculators will use the simplest formula.

Weber's formula gave strange answers at high velocity. Helmholtz integrated the trajectories of charged particles in a charged, insulator spherical shell, and demonstrated run-away solutions for speed. This was deemed a fatal flaw. However, Weber pointed out that Helmholtz' initial conditions were already super-luminal (tachyonic).

I want to emphasize that *all* of the equivalent Maxwell potentials such as Grassman, Biot-Savart, Whittaker and Weber, are non-relativistic.

Motivation for Re-examination

I like the ideas of purely electromagnetic mass, inertia based upon collective radiation reaction, and magnetic moments as an artifact of internal trajecties. I am re-examining historical force laws, looking for such solutions.

Low Speed Examination of Weber's Potential

The goal in this section is to accept the Weber formulations, and discuss them in the context of 1840-1870 electrical technology.

Verification of Engineering Equivalency

`http://www.kurtnalty.com/Weber.cpp` demonstrates the equivalence of Ampere, Biot-Savart, Whittaker and Weber force laws for forces involving closed current loops. A relevant snippet follows. Weber force calculations (F8) are done between source and sensor electrons, as well as between source electrons and sensor nucleons, and source nucleon and sensor electrons. The nucleon-nucleon calculation is not necessary for stationary coils.

```
//Ampere's Law
F4 += ir2*(2.0*dot(v1,v2) - 3.0*dot(v1,ar)*dot(v2,ar))*ar;

// Biot-Savart
F5 += ir2*(dot(v1,v2)*ar - dot(v2,ar)*v1);

//Whittaker Riemann
F6 += ir2*(dot(v1,v2)*ar - dot(v2,ar)*v1 - dot(v1,ar)*v2);

//Weber e- e-   proves inadequate
F7 += -ir2*(dot(V,V)-1.5*dot(ar,V)*dot(ar,V)+dot(R,A))*ar;

// Three interacting charge collections. Works
//Weber e- e-
F8 += -ir2*(dot(V,V)-1.5*dot(ar,V)*dot(ar,V)+dot(R,A))*ar;

//Weber p+ e-
F8-=-ir2*(dot(-v2,-v2)-1.5*dot(ar,-v2)*dot(ar,-v2)+dot(R,-a2))*ar;

//Weber e- p+
F8-=-ir2*(dot(v1,v1)-1.5*dot(ar,v1)*dot(ar,v1)+dot(R,a1))*ar;
```

As far as being consistent with existing observations, Weber's law gave the same engineering results as Ampere's or Biot-Savart laws, with only three times the previous effort. As a calculator, you would prefer to use Biot-Savart form, as this was the simplest formula of these.

Description of Terms

From the point of view of particle interactions, the Weber law is a modifed Gauss law for electric fields.

$$\vec{F} \;=\; \frac{q_i q_j}{4\pi\epsilon_0}\frac{\vec{r}_{ij}}{r_{ij}^3}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right)$$

The first term in parenthesis is the Coulomb force law. The second term provides a magnetic field correction which always works against the Coulomb term. The third term is the acceleration term, which can alternate from assisting to opposing the Coulomb term.

Because the force is purely radial, this force law cannot change angular momentum. This means that if we are not aimed directly at another charge, we will not collide, but rather orbit or scatter.

Closed Form Solutions with Elliptic Integrals

This section is based upon Assis, Chapter 7, with initial solution credited to Seegers in 1864.

Radial forces inherently preserve angular momentum.

$$\begin{aligned}
\vec{L} &= \vec{R} \times \vec{P} \\
&= m\vec{R} \times \vec{V} \\
\frac{d\vec{L}}{dt} &= m\vec{V} \times \vec{V} + m\vec{R} \times \vec{A} = 0
\end{aligned}$$

Since acceleration $\vec{A}$ is parallel to radial separation $\vec{R}$, both cross products zero out. Consequently, the angular momentum becomes a constant of motion, and the particle's orbit is in a plane.

For a particle in the plane with polar coordinates

$$\begin{aligned}
\vec{L} &= m\vec{R} \times \vec{V} \\
\vec{L} &= m\vec{R} \times (V_r\vec{a}_r + V_\theta\vec{a}_\theta) \\
\vec{L} &= m\vec{R} \times (\dot{r}\vec{a}_r + r\dot{\theta}\vec{a}_\theta) \\
\vec{L} &= m\vec{R} \times r\dot{\theta}\vec{a}_\theta \\
\vec{L} &= mr\vec{a}_r \times r\dot{\theta}\vec{a}_\theta \\
L_z &= mr(r\dot{\theta}) = mr^2\dot{\theta}
\end{aligned}$$

Place the origin at the center of mass. The two masses M_1 and M_2 are always connected by a line through this origin. If the separation between the two masses is r, then the two lengths as measured from the center of mass are

$$\begin{aligned}
r_1 &= r * M_2/(M_1 + M_2) \\
r_2 &= r * M_1/(M_1 + M_2) \\
r &= r_1 + r_2
\end{aligned}$$

Some authors, for example Bradbury *Theoretical Mechanics*, will use a negative sign on r_2 to emphasize that the two points are on opposite sides of the center of mass.

The total angular momentum is

$$\begin{aligned} L &= M_1 * r_1^2 * \dot{\theta} + M_2 * r_2^2 * \dot{\theta} \\ &= \dot{\theta}\left(M_1 * r * \frac{M_2}{M_1 + M_2} * r_1 + M_2 * r * \frac{M_1}{M_1 + M_2} * r_2\right) \\ &= \dot{\theta}\frac{M_1 M_2}{M_1 + M_2}\left(r * r_1 + r * r_2\right) \\ &= m r^2 \dot{\theta} \end{aligned}$$

with the reduced mass defined as

$$m = \frac{M_1 M_2}{M_1 + M_2}$$

The total energy for this system becomes

$$U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{q_1 q_2}{4\pi\epsilon r}\left(1 - \frac{\dot{r}^2}{2c^2}\right)$$

The first term above is the kinetic energy, and is always positive. The second term, the modified Coulomb potential, has an interesting behavior. When the radial separation speed exceeds $c\sqrt{2}$, we change from electrical repulsion to attraction, or vice versus. Choosing initial conditions with positive masses, positive charges, but high separation rates leads to a bound system (negative total energy), which I will address in the later section.

Under mundane conditions, positive energy means unbound systems. For example, repulsive charges, or the combination of attractive charges but higher kinetic energy. For inverse square force laws, negative energy is associated with a maximum radial separation (this being a turning point), and leading to the rule of thumb of bound systems.

Our roadmap for solving this problem begins with eliminating $\dot{\theta}$ from the energy equation using the angular momentum relationship above. Next, we change variables, leaving the time domain and instead finding an orbit equation $\theta(r)$. We determine maximum and minimum values for r, then find the angle corresponding to each radial value. Finally, knowing θ and r, we can integrate the angular momentum relationship for $\dot{\theta}$ to recover $\theta(t)$ and $r(t)$

Eliminate $\dot{\theta}$

$$\begin{aligned}\dot{\theta} &= \frac{L}{mr^2} \\ U &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{q_1 q_2}{4\pi\epsilon r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \\ &= \frac{m}{2}\left(\dot{r}^2 + \frac{L^2}{m^2 r^2}\right) + \frac{q_1 q_2}{4\pi\epsilon r}\left(1 - \frac{\dot{r}^2}{2c^2}\right)\end{aligned}$$

Find Turning Points

$$\begin{aligned}U &= \dot{r}^2\left(\frac{m}{2} - \frac{q_1 q_2}{4\pi\epsilon r}\frac{1}{2c^2}\right) + \frac{L^2}{2mr^2} + \frac{q_1 q_2}{4\pi\epsilon r} \\ \dot{r}^2 &= \frac{U - \frac{L^2}{2mr^2} - \frac{q_1 q_2}{4\pi\epsilon r}}{\frac{m}{2} - \frac{q_1 q_2}{8\pi\epsilon r c^2}}\end{aligned}$$

Turning points occur where the numerator equals zero.

$$\begin{aligned}0 &= U - \frac{L^2}{2mr^2} - \frac{q_1 q_2}{4\pi\epsilon r} \\ &= r^2 - r\frac{q_1 q_2}{4\pi\epsilon U} - \frac{L^2}{2mU} \\ r_{\text{turn}} &= \frac{q_1 q_2}{8\pi\epsilon U} \pm \sqrt{\left(\frac{q_1 q_2}{8\pi\epsilon U}\right)^2 + \frac{L^2}{2mU}}\end{aligned}$$

For $L = 0$, we get a collision as expected in the attractive case (charge signs differ), and a point of closest approach in the repulsive case (charge signs match) using the positive root.

For the next few cases, we need to be aware that the system energy can be positive, negative or zero.

When $U = 0$, we cannot determine a turning point, but we can say $Ur_{\text{turn}} = 0$, meaning r_{turn} is finite. Probably, this system does not evolve, but is static.

For $L \neq 0$, for the repulsive case (charge signs match and U is positive), we can use the positive root to find the radius of closest approach.

For $L \neq 0$, for the attractive case (charge signs differ), we have two cases. For bound systems, where $U < 0$, I believe we use the negative root. For unbound systems, where $U > 0$, we use the positive root.

Solving the Orbit Equation

Start with the energy equation. (We are still following the approach in Assis.)

$$\begin{aligned} U &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \quad \text{where} \\ \alpha &= \frac{q_1 q_2}{4\pi\epsilon} \end{aligned}$$

Change variables.

$$\begin{aligned} x^2 &= 1 - \frac{\alpha}{mc^2 r} \\ 2x\frac{dx}{dt} &= \frac{\alpha}{mc^2 r^2}\frac{dr}{dt} \\ \frac{dx}{dt} &= \frac{\alpha}{2mc^2 r^2 x}\frac{dr}{dt} \\ \frac{dx}{dt}\frac{dt}{d\theta} &= \frac{\alpha}{2mc^2 r^2 x}\frac{dr}{dt}\frac{dt}{d\theta} \\ \frac{dx}{d\theta} &= \frac{\alpha}{2mc^2 r^2 x}\frac{dr}{dt}\frac{mr^2}{L} \\ \frac{dr}{dt} &= \frac{2c^2 xL}{\alpha}\frac{dx}{d\theta} = \frac{c^2 L}{\alpha}\frac{d(x^2)}{d\theta} \\ r &= \frac{\alpha}{mc^2}\frac{1}{1 - x^2} \\ \frac{1}{r} &= \frac{mc^2(1 - x^2)}{\alpha} \\ \frac{\alpha}{r} &= mc^2(1 - x^2) \end{aligned}$$

Start substituting into the energy equation

$$\begin{aligned}
U &= \dot{r}^2 \left(\frac{m}{2} - \frac{q_1 q_2}{4\pi\epsilon r} \frac{1}{2c^2} \right) + \frac{L^2}{2mr^2} + \frac{q_1 q_2}{4\pi\epsilon r} \\
&= \dot{r}^2 \left(\frac{m}{2} - \frac{\alpha}{r} \frac{1}{2c^2} \right) + \frac{L^2}{2mr^2} + \frac{\alpha}{r} \\
U &= \left(\frac{2c^2 x L}{\alpha} \frac{dx}{d\theta} \right)^2 \left(\frac{m}{2} - \frac{\alpha}{r} \frac{1}{2c^2} \right) + \frac{L^2}{2mr^2} + \frac{\alpha}{r} \\
U &= \left(\frac{2c^2 x L}{\alpha} \frac{dx}{d\theta} \right)^2 \left(\frac{m}{2} - \frac{m(1-x^2)}{2} \right) + \frac{L^2}{2mr^2} + mc^2(1-x^2) \\
&= \left(\frac{2c^2 x L}{\alpha} \frac{dx}{d\theta} \right)^2 \left(\frac{mx^2}{2} \right) + \frac{L^2}{2mr^2} + mc^2(1-x^2) \\
&= \left(\frac{2c^2 x L}{\alpha} \frac{dx}{d\theta} \right)^2 \left(\frac{mx^2}{2} \right) + \frac{L^2}{2m} \left(\frac{mc^2(1-x^2)}{\alpha} \right)^2 + mc^2(1-x^2) \\
2\alpha^2 U &= 4c^4 x^4 L^2 m \left(\frac{dx}{d\theta} \right)^2 + L^2 \left(mc^4 (1-x^2)^2 \right) + 2\alpha^2 mc^2 (1-x^2)
\end{aligned}$$

Continuing after a short typesetting break,

$$4c^4 x^4 L^2 m \left(\frac{dx}{d\theta} \right)^2 = 2\alpha^2 U - L^2 mc^4 (1 - 2x^2 + x^4) - 2\alpha^2 mc^2 (1-x^2)$$

Divide both sides by $4c^4 x^4 L^2 m$

$$\begin{aligned}
\left(\frac{dx}{d\theta} \right)^2 &= \frac{2\alpha^2 (U/m) - L^2 c^4 (1 - 2x^2 + x^4) - 2\alpha^2 c^2 (1-x^2)}{4c^4 x^4 L^2} \\
&= \frac{x^4(-L^2 c^4) + x^2(2L^2 c^4 + 2\alpha^2 c^2) + (2\alpha^2 (U/m) - L^2 c^4 - 2\alpha^2 c^2)}{4c^4 x^4 L^2} \\
&= \frac{x^4(-c^4) + x^2(2c^4 + 2\alpha^2 c^2 L^{-2}) + (2\alpha^2 L^{-2} (U/m) - c^4 - 2\alpha^2 L^{-2} c^2)}{4c^4 x^4} \\
&= \frac{x^4(-1) + x^2(2 + 2\alpha^2 c^{-2} L^{-2}) + (2\alpha^2 L^{-2} c^{-4} (U/m) - 1 - 2\alpha^2 L^{-2} c^{-2})}{4x^4}
\end{aligned}$$

We can de-clutter a bit by introducing

$$\begin{aligned}
\beta^2 &= \frac{\alpha^2}{L^2c^2} \\
\left(\frac{dx}{d\theta}\right)^2 &= \frac{x^4(-1) + x^2(2+2\beta^2) + (2\beta^2c^{-2}(U/m) - 1 - 2\beta^2)}{4x^4} \\
&= \frac{x^4(-1) + x^2(2+2\beta^2) + (-1 - 2\beta^2\left(1 - (U/\left(mc^2\right))\right))}{4x^4}
\end{aligned}$$

Let's factor the numerator.

$$\begin{aligned}
0 &= x^4 + x^2(-2-2\beta^2) + \left(1 + 2\beta^2\left(1 - \left(U/\left(mc^2\right)\right)\right)\right) \\
x^2 &= (1+\beta^2) \pm \sqrt{(1+\beta^2)^2 - (1 + 2\beta^2\left(1 - (U/\left(mc^2\right))\right))} \\
&= (1+\beta^2) \pm \sqrt{\beta^4 + (2\beta^2U/(mc^2))}
\end{aligned}$$

Define

$$\begin{aligned}
x_A^2 &= (1+\beta^2) + \sqrt{\beta^4 + (2\beta^2U/(mc^2))} \\
x_B^2 &= (1+\beta^2) - \sqrt{\beta^4 + (2\beta^2U/(mc^2))}
\end{aligned}$$

Then, our differential equation is

$$\begin{aligned}
\frac{dx}{d\theta} &= \pm\frac{1}{2x^2}\sqrt{(x_A^2 - x^2)(x^2 - x_B^2)} \\
\frac{d\theta}{dx} &= \pm\frac{2x^2}{\sqrt{(x_A^2 - x^2)(x^2 - x_B^2)}} \\
\theta(x) &= \pm\int_x^{x_A} \frac{2x^2dx}{\sqrt{(x_A^2 - x^2)(x^2 - x_B^2)}} \\
&= \pm 2x_A E(\phi, k)
\end{aligned}$$

by Gradshteyn, p247 formula 3.153.8, where

$$\begin{aligned}
\phi &= \sin^{-1}\left(\sqrt{\frac{x_A^2 - x^2}{x_A^2 - x_B^2}}\right) \\
k &= \sqrt{\frac{x_A^2 - x_B^2}{x_A^2}}
\end{aligned}$$

Calculating and Plotting an Orbit

So, now that we have a formula, how do we get an orbit, or better still, a time history?

The program WeberForces.c, at `http://www.kurtnalty.com/WeberForces.c`, which uses `http://mymathlib.webtrellis.net/c_source/functions/elliptic_integrals/legendre_elliptic_integral_second_kind.c` solves for the orbit of an electron in a hydrogen atom environment at a potential slightly above ground level.

Given initial conditions of charges, masses, positions and velocities, we calculate a reduced mass and center of gravity. We calculate some constants of convenience, $\alpha = (q_1 q_2)/(4\pi\epsilon)$ and $\beta^2 = (\alpha^2)/(L^2 c^2)$. We next calculate two limiting parameters, x_A^2 and x_B^2. The two parameters x_A^2 and x_B^2 above provide our information about the minimum and maximum radius. These radii match the previous rmin and rmax from the energy equation.

```
rmin = ((alpha)/(m*c*c))*((1.0)/(1.0-xA2));
rmax = ((alpha)/(m*c*c))*((1.0)/(1.0-xB2));
```

We also get the angular extent for this semi-orbit.

```
phi = asin(sqrt((xA2-xB2)/(xA2-xB2)));  // asin(1) = 90 degrees
theta_high = 2.0*sqrt(xA2)
    *Legendre_Elliptic_Integral_Second_Kind(phi,'k',k);

phi = asin(sqrt((xA2-xA2)/(xA2-xB2)));  // asin(0) = 0 degrees
theta_low = 2.0*sqrt(xA2)
    *Legendre_Elliptic_Integral_Second_Kind(phi,'k',k);
```

For the example studied, Theta_high = 180.005 degrees, and Theta_low = 0 degrees. We see the precession amount at this energy and eccentricity is quite small.

For the simulation, I start with the lower value of xB2, calculate R from x2, then calculate phi and theta from x2. I then step 1% toward xA2, and repeat until we reach xA2.

```
output = fopen("R_theta_x_y.txt","w");
dx2 = 0.01*(xA2 - xB2); // increment 1%
for (i=0;i<101;i++) {
```

```
    x2 = xB2 + i*dx2;
    rFromX = ((alpha)/(m*c*c))*((1.0)/(1.0-x2));

    phi = asin(sqrt((xA2-x2)/(xA2-xB2)));
    theta = 2.0*sqrt(xA2)*
       Legendre_Elliptic_Integral_Second_Kind(phi,'k',k);

    fprintf(output,"%g  %g  %g   %g   \n",rFromX,
       theta, rFromX*cos(theta), rFromX*sin(theta) );
}
fclose(output);
```

The output file has four columns, being radius, radian angle, x coordinate and y coordinate. I used the first two columns to plot a radial impulse diagram for this simulation using GnuPlot

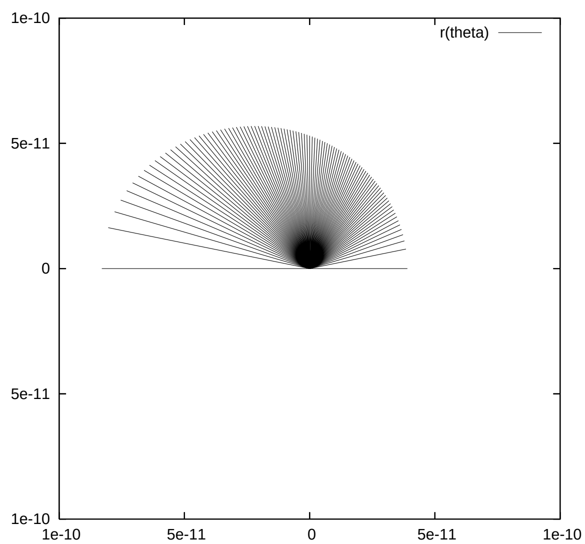
```
Plot with GnuPlot - commands:

set term postscript
set output "R_theta.ps"
set polar
set size square
set xrange[-1e-10:1e-10]
set yrange[-1e-10:1e-10]
plot "R_theta_x_y.txt" using 2:1 title "r(theta)" with impulses
```

The postscript image is turned into a pdf using ps2pdf, then included as Figure 1.

Bypassing x^2 and using r

Given the simple relations between x^2 and r, and the especially simple answer format, I now eliminate the use of x^2, and simply present the same results parameterized in terms of r.

Figure 1 - Radius versus Angle for Electron Near Proton



$$
\begin{aligned}
\alpha &= \frac{q_1 q_2}{4\pi\epsilon} \\
\beta^2 &= (\alpha^2)/(L^2 c^2) \\
U &= \frac{m}{2}(\dot{r}^2 + \frac{L^2}{m^2 r^2}) + \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \\
r_{min} &= \frac{\alpha}{2U} - \sqrt{\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU}} \\
r_{max} &= \frac{\alpha}{2U} + \sqrt{\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU}}
\end{aligned}
$$

Define

$$
\rho = -\frac{\alpha}{mc^2}
$$

This number just happens to be the classical electron radius. Using ρ and r, we can write the expressions for k, ϕ and θ.

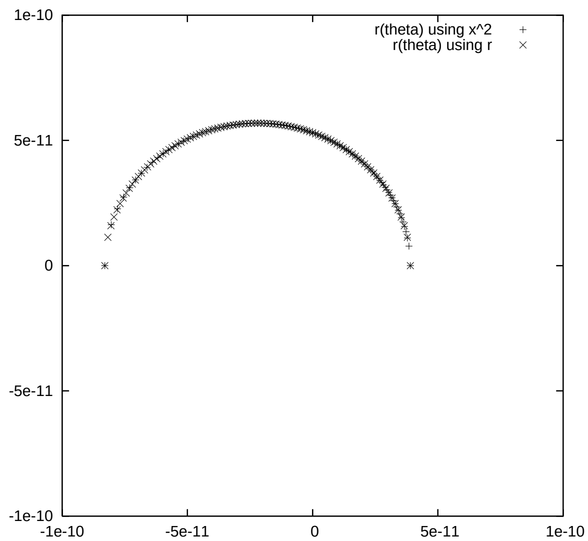
$$
\begin{aligned}
k &= \sqrt{\frac{\rho(r_{max} - r_{min})}{r_{max}(r_{min} + \rho)}} \\
\phi(r) &= \sin^{-1}\sqrt{\frac{1 - (r_{min}/r)}{1 - (r_{min}/r_{max})}} \\
\theta(r) &= \pm 2\sqrt{1 + (\rho/r_{min})}E(\phi, k)
\end{aligned}
$$

The agreement between these two formula sets is shown in Figure 2.

The above formulas have a nice geometrical interpretation. $E(\phi, k)$ is the distance travelled along an ellipse of unit major radius, travelling to angle ϕ starting at the minor radius $\sqrt{1-k^2}$. The scale factor in from of E is can be a scaling factor for the ellipse, or the rolling circle, or split among both. A unit rolling circle covers an angle θ during this event. Of course, if we are driving θ, then the ellipse rolls correspondingly.

Now, the next step in making this simulation user friendly, is to invert $\theta(r)$ to provide $r(\theta)$. Getting ϕ from $E(\phi, k)$ seems to be a task that just hasn't been widely done. This inversion can be done graphically, by rotating the image of $E(\phi, k)$ around the 45 degree diagonal. This inversion can probably

Figure 2 - X,Y Overlay for R versus x^2 Calculations



be done by inverting the series expansion. Whenever I figure this out, I'll revise and finish this section.

At any rate, now that we've verified reasonable behavior at low speeds, let go to outrageous speeds next!

Science Fiction Aspects of Weber's Potential

The goal in this section is to continue to literally accept the Weber formulations, and look at microscopic and transluminal extrapolations as well as Boscovitch-like behavior.

Roger Joseph Boscovich, also known as Ruggero Giuseppe Boscovich (1711-1787) proposed a force law for solids which is attractive at large distances, yet turns repulsive or even oscillatory at close range. He also discussed variations, where the repulsive forces can arise due to velocity or acceleration terms between the particles.

Coulomb's law for electrostatics, $U = q_1 q_2/(4\pi\epsilon)$, is alway repulsive or attractive depending upon the charges involved. Biot-Savart's non-relativistic extensions for magnetics, $U = q_1 q_2(1 - \vec{v}_1 \cdot \vec{v}_2/c^2)/(4\pi\epsilon)$, reduces the Coulomb force between parallel currents at low speeds, nulls out the Coulomb for at light speed, and overwhelms repulsion at superluminal speeds.

Weber's potential, $U = q_1 q_2(1 - \dot{r}^2/2c^2)/(4\pi\epsilon)$, by contrast, reduces the Coulomb force between approaching/retreating charges, rather than parallel charges. This magnetic term does nulls out, then overwhelms, the Coulomb force as the radial speed exceeds 1.414 c.

Weber's Potential with Angular Momentum Substitution

In reduced mass, center of gravity coordinates, we have the Weber energy as

$$\begin{aligned}
U &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \quad \text{where} \\
\alpha &= \frac{q_1 q_2}{4\pi\epsilon} \\
L &= mr^2\dot{\theta} \quad \text{so} \\
r\dot{\theta} &= \frac{L}{mr} \\
U &= \frac{m}{2}(\dot{r}^2 + \frac{L^2}{m^2r^2}) + \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \\
U &= \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \frac{1}{r}\frac{\alpha}{2c^2}\dot{r}^2
\end{aligned}$$

Superluminal Bonding for Like Polarity Charges

Look at a case where we want two negative charges to be bound. This can be done with the Weber force law if we make the system energy negative. Looking at the first three terms on the right hand side, we have strictly positive quantities. The only chance for a negative contribution comes from the fourth term, which incidently does not care about approach or retreat. We see that the *minimum* radial speed for attraction is $\dot{r} > \sqrt{2}c$.

Let's rearrange this equation looking at $\dot{r}^2$.

$$\begin{aligned}
U &= \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \frac{1}{r}\frac{\alpha}{2c^2}\dot{r}^2 \\
\dot{r}^2 &= \frac{U - \frac{L^2}{2mr^2} - \frac{\alpha}{r}}{\frac{m}{2} - \frac{1}{r}\frac{\alpha}{2c^2}}
\end{aligned}$$

The left hand side of the equation, as a square, is always positive. The numerator has three always negative terms, and so is always negative. Consequently, we require the denominator to be always negative. This in turn, sets a maximum radius for this system.

$$\begin{aligned} \frac{m}{2} - \frac{1}{r}\frac{\alpha}{2c^2} &< 0 \\ \frac{m}{2} &< \frac{1}{r}\frac{\alpha}{2c^2} \\ r &< \frac{\alpha}{mc^2} \end{aligned}$$

Special Case Where $L = 0$

For the special case where $L = 0$, as $r \to 0$, our expression for $\dot{r}^2$ limits to

$$\begin{aligned} \dot{r}^2 &= \frac{U - \frac{\alpha}{r}}{\frac{m}{2} - \frac{1}{r}\frac{\alpha}{2c^2}} \\ &= \frac{2rc^2U - 2\alpha c^2}{mrc^2 - \alpha} \\ &= 2c^2 \quad \text{when r} \ = \ 0 \end{aligned}$$

We have the interesting situation, where we hit our minimum speed possible at $r = 0$. In effect, we slow down to just above $1.414c$ as we hit zero. We now ask, what happens next? A simple minded approach indicates that r now goes negative, and this, in turn, has profound consequences for our potentials and forces.

In the expressions above, we have $\alpha/r = q_1q_2/(4\pi\epsilon r)$. The consequence of r going negative is indistinguishable from one of the charges changing polarity. We now have a new simulation, of opposite charges at very close separation, and a separation speed at $1.414c$.

Let's return to the energy equation, for $L = 0$.

$$\begin{aligned} U &= \frac{m}{2}\dot{r}^2 + \frac{\alpha}{r} - \frac{1}{r}\frac{\alpha}{2c^2}\dot{r}^2 \\ U - \frac{m}{2}\dot{r}^2 &= \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \end{aligned}$$

The left hand side is always negative. For the $r > 0$ case, to get the right side negative required $\dot{r}^2 > 2c^2$. Now, for the $r < 0$ case, we require just the opposite, namely $\dot{r}^2 < 2c^2$, and slowing down. How far out do we go before hitting a turning point? Setting $\dot{r} = 0$,

$$\begin{aligned} U &= \frac{\alpha}{r} \\ r &= \frac{\alpha}{U} \quad \text{maximum negative radius} \end{aligned}$$

After hitting this turning point, we reverse, and accelerate heading back toward zero. We hit $r = 0$ at $1.414c$, as before, and now are outbound, positive radius, and accelerating.

We know from previously, that there will be a maximum radial separation. This system is going to have a hard bounce at $r_{max} = \alpha/(mc^2)$. How fast will we be at the hard bounce point? We are going infinitely fast due to the divide by zero. These results are similar in spirit to tachyon calculations, where instead of a critical speed c, we have a critical speed $c\sqrt{2}$. For tachyons, minimum kinetic energy occurs at infinite speed, and lower, finite speeds are a higher kinetic energy.

Knowing the overall behavior, let's now solve the radial velocity differential equation. As an added bonus, since $L = 0$, this is also the speed and velocity equations as well.

$$\begin{aligned} \dot{r}^2 &= \frac{2rc^2U - 2\alpha c^2}{mrc^2 - \alpha} \\ &= \frac{2}{m}\left[\frac{rc^2U - \alpha c^2}{rc^2 - \alpha/m}\right] \\ &= \frac{2}{m}\left[\frac{rU - \alpha}{r - \alpha/(mc^2)}\right] \end{aligned}$$

Define

$$\rho = \frac{\alpha}{mc^2}$$

Then

$$\begin{aligned} \dot{r}^2 &= \frac{2}{m}\left[\frac{rU - \alpha}{r - \rho}\right] \\ &= \frac{2}{m}\left[\frac{rU - \rho U + \rho U - \alpha}{r - \rho}\right] \\ &= \frac{2}{m}\left[U + \frac{\rho U - \alpha}{r - \rho}\right] \end{aligned}$$

The constant ρ is the maximum separation in the positive radial case we saw earlier. I'll call this distance "the wall", as it is the place of reflection in the high speed case. We see when $r = \rho$, we have our infinite speed case. We also see when $r = \alpha/U$, (a negative radius, by the way), that $\dot{r} = 0$, as before.

We now make a change of coordinates. Let $x = \rho - r$ be the distance from the wall. This x will be a positive number between $x_{min} = 0$ (the wall) and $x_{max} = \rho - U/\alpha$. During most of this time, r is negative. Keep in mind that ρ is a constant.

$$\begin{aligned}
\dot{r}^2 &= \frac{2}{m}\left[U + \frac{\rho U - \alpha}{r - \rho}\right] \\
\dot{r}^2 &= \frac{2}{m}\left[U + \frac{\alpha - \rho U}{\rho - r}\right] \\
\left(\frac{d(\rho - r)}{dt}\right)^2 &= \frac{2}{m}\left[U + \frac{\alpha - \rho U}{\rho - r}\right] \\
\dot{x}^2 &= \frac{2}{m}\left[U + \frac{\alpha - \rho U}{x}\right] \\
&= \frac{2U}{m} + \frac{2(\alpha - \rho U)}{m}\frac{1}{x}
\end{aligned}$$

Define

$$\begin{aligned}
a &= -\frac{2U}{m} \quad \text{U always negative so a is positive} \\
b &= \frac{2(\alpha - \rho U)}{m} \quad \text{always positive}
\end{aligned}$$

Our differential equation is then

$$\begin{aligned}
\dot{x}^2 &= -a + (b/x) \\
\dot{x}^2 &= (b/x) - a \\
\frac{dx}{dt} &= \sqrt{(b/x) - a} \\
dt &= \frac{dx}{\sqrt{(b/x) - a}} \\
&= \frac{\sqrt{x}dx}{\sqrt{b - ax}} \\
t &= \int_0^x \frac{\sqrt{y}}{\sqrt{b - ay}}dy
\end{aligned}$$

Substitute

$$
\begin{aligned}
y &= \frac{b}{a}\sin^2\theta \\
t &= \int_0^x \frac{dy}{\sqrt{(b/y)-a}} \quad \text{becomes} \\
t &= \frac{b}{a\sqrt{a}}\int_0^{\phi}(1-\cos(2\theta))\,d\theta \\
\phi &= \sin^{-1}\left(\sqrt{\frac{xa}{b}}\right) \\
t &= \frac{b}{a\sqrt{a}}\left[\phi - \frac{1}{2}\sin(2\phi)\right]
\end{aligned}
$$

If you want a longer formula without trig substitutions,

$$
t = \frac{b}{a\sqrt{a}}\left[\sin^{-1}\left(\sqrt{\frac{xa}{b}}\right) - \sqrt{\frac{xa}{b}}\sqrt{1-\frac{xa}{b}}\right]
$$

So, now that we have our formulas, let's plot $x(t)$ and $v(t)$. The file "SimulateTwoWeberBoundElectronsAt-13.6V.c", available at `http://www.kurtnalty.com/SimulateTwoWeberBoundElectronsAt-13.6V.c` models two superluminally bound electrons.

We have two regimes. From $x = 0$ to $x = \rho$ we are inbound from infinite speed having hit the wall at $x = 0$. We have two electrons, both with positive radial separation, at very superluminal speeds. This regime only lasts 8.68721e-24 seconds, and only covers 5.55642e-15 meters. We drop from infinite speed to 4.26402e+08 ($c\sqrt{2}$) m/s during this time. Figure 3 provides distance versus time, while Figure 4 provides speed versus time for this regime. I suspect outside observers would see two negative electrons during this very short interval.

The next regime goes from $x = \rho$ to $x = \rho - r_{max}$, which is the negative radial separation regime. This regime lasts a much longer 4.69246e-17 seconds, and covers a huge 1.04692e-10 meters. We drop from 4.26402e+08 ($c\sqrt{2}$) m/s to zero during this time. Figure 4 provides distance versus time, while Figure 6 provides speed versus time for this regime. I suspect outside observers could not distinguish what they see from an electron and a positron during this interval, excepting the difficulty of assigning charge to

Figure 3 - Inbound Position Versus Time

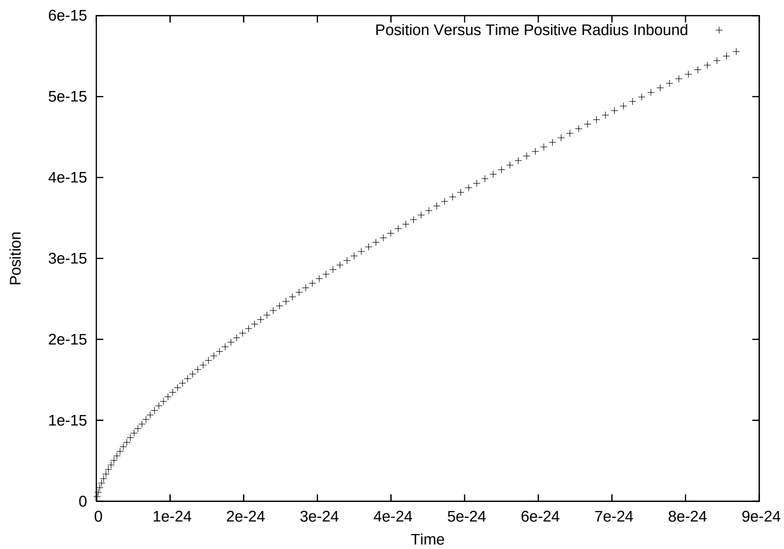


Figure 4 - Inbound Velocity Versus Time

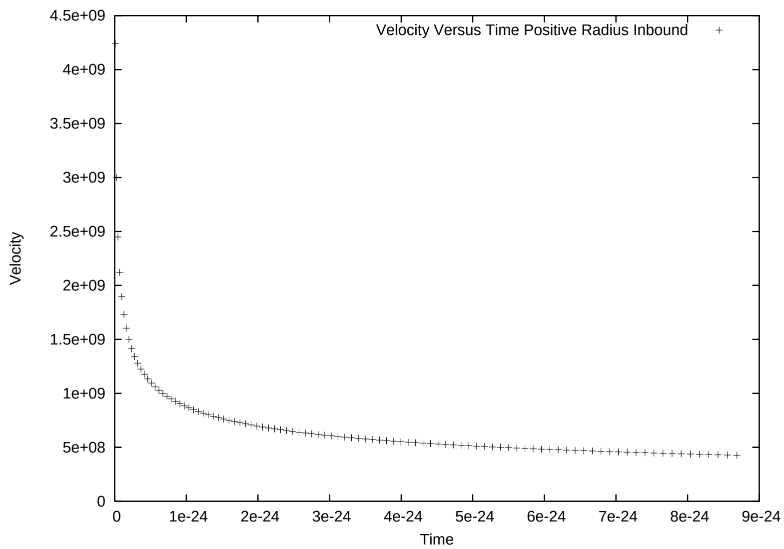
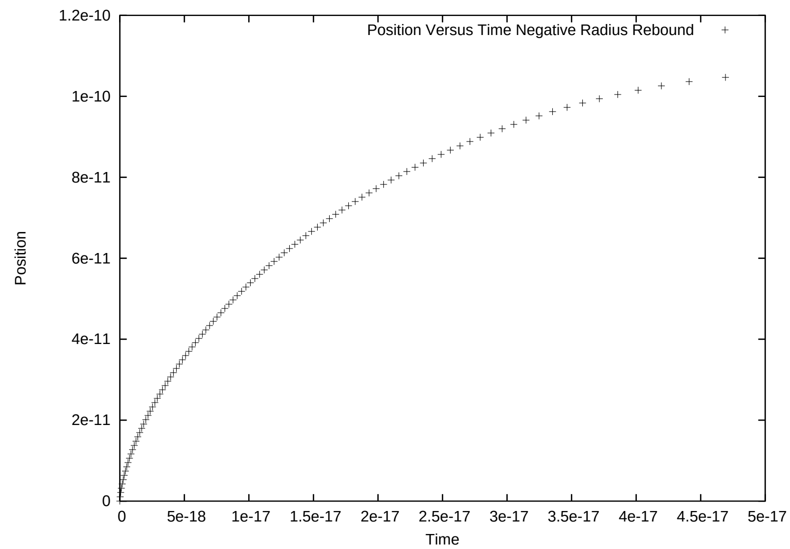


Figure 5 - Outbound Position Versus Time



the individual elements. I suspect the outside world would see a neutral system during this time. (The third particle observor simulations are another priotity task.)

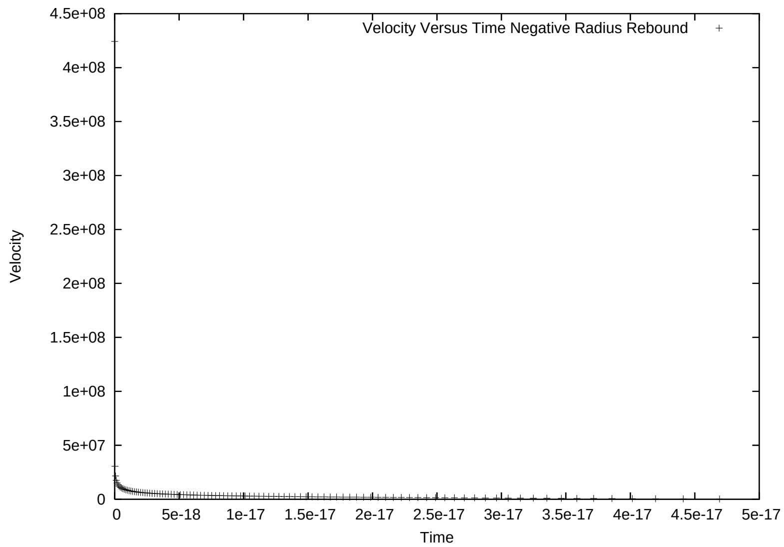
General Discussion of $L = 0$ Case

As the binding energy increases, the $r_{max} = \alpha/U$ distance on the negative radial rebound decreases. Our system shrinks with increasing binding energy. From the formulas $\rho = \alpha/(mc^2)$ and $r_{max} = \alpha/U$, we see that the magnitude of the wall bounce equals the negative radial excursion when the magnitude of the bound energy is mc^2. This may be significant.

The run-away solutions of Helmholtz, in my opinion, are a lost opportunity of their time. I am perfectly happy with tachyonic behavior, where kinetic energy decreases with velocity, and have no problem with reflection at the maximal radius, since the equations give $\dot{r}^2$, which is conserved on reflection through infinite speed (the zero kinetic energy speed for tachyons).

So, summarizing, given a negative system energy, two equal polarity

Figure 6 - Outbound Velocity Versus Time



charges oscillate along a line through their centers. Regardless of initial energy, the speed at impact is $c\sqrt{2}$. The motion has two regimes, corresponding to positive and negative radial separations. In the positive radial separation mode, the charges separate at $c\sqrt{2}$ at collision, rapidly accelerate to infinite speed at a distance $\rho = \alpha/(mc^2)$. At this distance, their tachyonic kinetic energy is zero, and they reflect back toward the origin. During the fall, they decelerate from infinite speed down to $c\sqrt{2}$ at collision. They pass through each other at this speed, then have a change in their force law due to the negative radial separation ala CPT. They continue to decelerate to zero speed at $r_{max} = \alpha/U$, then reverse direction at this point of zero ordinary kinetic energy, and the cycle repeats.

General Case where $L \neq 0$

When $L \neq 0$, we won't have a collision, and there will be a minimum radius of approach, as well as a maximum. The biggest difference between this case and the previous system should be an absense of negative radial motion.

We begin by looking at the energy equation including angular momentum, and solve for the turning points.

$$\begin{aligned} U &= \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\alpha}{r} - \frac{1}{r}\frac{\alpha}{2c^2}\dot{r}^2 \\ \dot{r}^2 &= \frac{Ur^2 - \frac{L^2}{2m} - \alpha r}{\frac{m}{2}r^2 - r\frac{\alpha}{2c^2}} = 0 \quad \text{implies} \\ 0 &= Ur^2 - \frac{L^2}{2m} - \alpha r \\ 0 &= r^2 - \frac{\alpha r}{U} - \frac{L^2}{2Um} \\ r_{turn} &= \frac{\alpha}{2U} \pm \sqrt{\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU}} \end{aligned}$$

We see we have two turning points (no surprise). Given that the system is bound, meaning $U < 0$, the square root term will be less in magnitude than the magnitude of $\alpha/(2U)$. For the case of an attractive force between opposite charges, where $\alpha < 0$ and $\alpha/(2U) > 0$, we get a reasonable set of positive radii similar to the low speed example dealt with in the first part of this paper.

So, let's look the turning points for the case of similar charges with $L \neq 0$. Because $\alpha > 0$ due to both charges being the same polarity, given $U < 0$, we have two negative radii. In essense, if I demand a bound system with similar charges, the mathematics cheerfully reply back 'Quite Right Sir! We hope you are happy with negative radii which are indistinguishable from a reversed charge.'

A numerical illustration is useful. The left column is a positron/electron combo, while the right column is an electron/electron combo at the same energy and momentum.

$$
\begin{array}{rcl@{\qquad}rcl}
U &=& -2.17610^{-18} & U &=& -2.17610^{-18} \\
m &=& 4.55510^{-31} & m &=& 4.55510^{-31} \\
q1 &=& -1.610^{-19} & q1 &=& -1.610^{-19} \\
q2 &=& 1.610^{-19} & q2 &=& -1.610^{-19} \\
\alpha &=& -2.3008610^{-28} & \alpha &=& 2.3008610^{-28} \\
L_{\text{Max}} &=& 7.4437310^{-35} & L_{\text{Max}} &=& 7.4437310^{-35} \\
L &=& 5.9549810^{-35} & L &=& 5.9549810^{-35} \\
a &=& 5.2869110^{-11} & a &=& -5.2869110^{-11} \\
r1 &=& 8.4590510^{-11} & r1 &=& -2.1147610^{-11} \\
r2 &=& 2.1147610^{-11} & r2 &=& -8.4590510^{-11}
\end{array}
$$

We can't distinguish between bound opposite charges, or reverse bound similar charges. As a consequence, I will only look at the opposite charge case.

Angular Momentum and Velocity

We see another interesting feature from the turning point equation. To keep the terms under the square root positive, we will have a maximum possible angular momentum for each energy level. This is not too surprising. Thinking about closed orbits. If we increase L too much, we do escape the bound system and scatter. If we are at a known energy level, there will be a maximum angular momentum for orbiting, rather than scattering.

$$\begin{aligned}
\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU} &> 0 \\
\frac{L^2}{2mU} > -\left(\frac{\alpha}{2U}\right)^2 \\
L^2 < -2mU\left(\frac{\alpha}{2U}\right)^2 = -\frac{m\alpha^2}{2U} \\
L_{max} < \sqrt{-\frac{m\alpha^2}{2U}}
\end{aligned}$$

When we are at the turning points, $\dot{r} = 0$, and our velocity at that instant is $v = r\dot{\theta} = L/(mr)$. Since L_{max} and $r_{min} = r2$ are known, we can calculate the maximum speed expected.

$$\begin{aligned}
v_{max} &= r\dot{\theta} \\
&= \frac{L}{mr_2} \quad \text{where} \\
r_2 &= \frac{\alpha}{2U} - \sqrt{\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU}}
\end{aligned}$$

We now want to look at the special case where $L = L_{max}$. At this energy, we have a circular orbit of radius $r = \alpha/(2U))$.

$$\begin{aligned}
L &= \sqrt{-\frac{m\alpha^2}{2U}} \\
r &= \frac{\alpha}{2U} \\
v &= \frac{L}{mr} = \sqrt{-\frac{2U}{m}}
\end{aligned}$$

We can achieve any speed we wish simply by cranking down the binding energy. Nothing prevents superluminal behavior in this slowest of cases.

The more general case has $L < L_{max}$. In this more general case, we will find peak speeds faster the circular case for the same energy. Let $a = L/L_{max}$. Then

$$
\begin{aligned}
r_2 &= \frac{\alpha}{2U} - \sqrt{\left(\frac{\alpha}{2U}\right)^2 + \frac{L^2}{2mU}} \\
&= \frac{\alpha}{2U}\left(1 - \sqrt{1-a^2}\right) \\
v &= \frac{L}{mr} = \frac{aL_{max}}{\frac{m\alpha}{2U}\left(1-\sqrt{1-a^2}\right)} \\
&= \frac{a2U\sqrt{-\frac{m\alpha^2}{2U}}}{m\alpha\left(1-\sqrt{1-a^2}\right)} \\
&= \frac{a}{\left(1-\sqrt{1-a^2}\right)}\sqrt{\frac{-2U}{m}} \\
&= \frac{1+\sqrt{1-a^2}}{a}\sqrt{\frac{-2U}{m}}
\end{aligned}
$$

The forefactor starts at one when $a = 1$, and shoots to infinity as $a \to 0$. Consequently, we can achieve any high speed we wish by choosing an energy level, then choosing a low enough angular momentum. Notice the trick with the forefactor on the last two lines above. We will use this below in a bit.

Examine the kinetic and potential energies of this orbiting system at minimum separation. We know the separation from above. We know $\dot{r} = 0$ at minimaxes, and we know v at this minimum.

$$
\begin{aligned}
KE &= \frac{1}{2}mv^2 \\
&= \frac{m}{2}\left(\frac{1+\sqrt{1-a^2}}{a}\right)^2 \frac{-2U}{m} \\
&= \frac{m}{2}\left(\frac{1+\sqrt{1-a^2}}{a}\right)\left(\frac{a}{1-\sqrt{1-a^2}}\right)\frac{-2U}{m} \quad \text{notice the trick?} \\
&= -U\left(\frac{1+\sqrt{1-a^2}}{1-\sqrt{1-a^2}}\right) \\
&= -U\left(\frac{2+2\sqrt{1-a^2}-a^2}{a^2}\right) \qquad \text{or the initial approach}
\end{aligned}
$$

$$
\begin{aligned}
PE &= \alpha \frac{1}{r} \\
&= \alpha \frac{1}{\frac{\alpha}{2U}\left(1-\sqrt{1-a^2}\right)} \\
&= \frac{2U}{\left(1-\sqrt{1-a^2}\right)} \\
U &= PE + KE
\end{aligned}
$$

Nicely enough, none of the high speed cases requires any modification of the previous closed form solutions for closed orbits. Definitely, there will be greater precession at higher speeds. However, no special treatment is needed for superluminal orbital solutions.

Speculation About the Capture Process

I am fairly sensitive toward the details of electron capture requiring maximized angular momentum for the bound states. In my mind's eye, I perceive a 'bumper cars' event, where an electron, which happens to be marginally above capture angular momentum scatters initially from the target. However, radiation reaction brings the two together again at a closer match, until coupling occurs *just at* the maximum possible angular momentum.

Speculation About Energy Limits

All of this work has been a non-relativistic treatment which does not limit speed.

However, we are fairly sure that speeds have a pole at c. If we ignore increasing effective mass effects, we can see that limiting $v = c$ will provide a limit on angular momentums available, and provide a basement for bound energies.

As an example, from the previous Weber equation work,

$$
\begin{aligned}
L_{\max} &= \sqrt{-\frac{m\alpha^2}{U}} = mrv \\
&= m\left(\frac{\alpha}{2U}\right)c
\end{aligned}
$$

This implies

$$\begin{aligned}
\sqrt{-\frac{m\alpha^2}{U}} &= m\left(\frac{\alpha}{2U}\right)c \\
\sqrt{-mU\alpha^2} &= \frac{m\alpha}{2}c \\
\sqrt{-mU} &= \frac{m}{2}c \\
U &= (mc^2)/4
\end{aligned}$$

This corresponds to a base energy level of 256 keV for positronium, which is much higher than measure values. However, the principle that relativistic speed limits can limit angular momentum does have merit, in my opinion.

Commentary About this Exercise

The value of this exercise has been extrapolation of a massive, low speed force law candidate to a high speed regime. While the Weber law does not satisfy our relativistic experience, it does demonstrate Boscovitch style modifications of forces from attractive to repulsive based upon some criterion other than separation. To provide a relativistic force, at a minimum, we need to include the Leonard Wiechert corrections for apparent charge, and provide field calculations bases upon retarded images of the source particles.

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