Useful Elliptic Integral Formulas for Magnetics

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Abstract

Complete elliptic integrals naturally arise when calculating inductances, magnetic potentials and magnetic fields using circular and cylindrical coils. Specifically, potentials include terms divided by separation. This separation involves, via the law of cosines, constants and cosines under the square root. When the sources are circular paths, closed loop integrals naturally occur.

This note provides a quick list of useful elliptic integral relationships. I then provide a short bit of code to calculate the elliptic integrals using C. I finish by providing step by step derivations of some of these formulas as worked out examples.

Standard Form with Dialects

We have three fundamental elliptic integrals of the first, second and third kinds, each of which has three incompatible dialects regarding the parameters, being the m, k and alpha forms. Check the defining integral being used to prevent errors in the value of the arguments due to the similar but different definitions.

Complete elliptic integrals are a subset of the related elliptic integral where the parameter has been set to the maximum value possible.

Here are the most common definitions and dialects.

Elliptic Integrals of the First Kind

$$
\begin{aligned}
F(x|m) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \\
F(x,k) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\
F(\phi\backslash\alpha) &= \int_0^\phi \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}}
\end{aligned}
$$

Complete Elliptic Integrals of the First Kind

$$
\begin{aligned}
K(m) = F(1|m) &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \\
K(k) = F(1,k) &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\
K(\alpha) = F(\frac{\pi}{2}\backslash\alpha) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}}
\end{aligned}
$$

Elliptic Integrals of the Second Kind

$$
\begin{aligned}
E(x|m) &= \int_0^x \sqrt{\frac{(1-mt^2)}{(1-t^2)}}dt \\
E(x,k) &= \int_0^x \sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt \\
E(\phi\backslash\alpha) &= \int_0^\phi \sqrt{1-\sin^2\alpha\sin^2\theta}d\theta
\end{aligned}
$$

Complete Elliptic Integrals of the Second Kind

$$
\begin{aligned}
E(m) = E(1|m) &= \int_0^1 \sqrt{\frac{(1-mt^2)}{(1-t^2)}}dt \\
E(k) = E(1,k) &= \int_0^1 \sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt \\
E(\alpha) = E(\frac{\pi}{2}\backslash\alpha) &= \int_0^{\pi/2} \sqrt{1-\sin^2\alpha\sin^2\theta}d\theta
\end{aligned}
$$

Elliptic Integral of the Third Kind

$$
\Pi(n;\phi\backslash\alpha) = \int_0^{\phi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-\sin^2\alpha\sin^2\theta}}
$$

Complete Elliptic Integral of the Third Kind

$$
\Pi(n;\frac{\pi}{2}\backslash\alpha) = \int_0^{\pi/2} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-\sin^2\alpha\sin^2\theta}}
$$

In all of these forms, the integrals are even functions of the integrating variable due to the square terms in the arguments. We will use this fact below, as we change to some less commonly listed variations.

Useful Formulas

Most of these formulas can be verified in Gradshteyn and Rhyzik. Others have been derived by simple substitutions, such as $\sin^2\theta = 0.5 - 0.5 * \cos(2\theta)$ and subsequent rework of variables. The loop integrals formulas have exploited the arbitrariness of starting phase to provide multiple representations. In all cases, these formulas are the ones I wish were tabulated, as they arise naturally when working with current loops.

$$\oint \sqrt{1 \pm k^2 \cos\theta} d\theta = 4\sqrt{1+k^2} E\left(\sqrt{\frac{2k^2}{1+k^2}}\right)$$

$$\oint \sqrt{1 \pm k^2 \sin\theta} d\theta = 4\sqrt{1+k^2} E\left(\sqrt{\frac{2k^2}{1+k^2}}\right)$$

$$\oint \frac{d\theta}{\sqrt{1 \pm k^2 \cos\theta}} = \frac{4}{\sqrt{1+k^2}} K\left(\sqrt{\frac{2k^2}{1+k^2}}\right)$$

$$\oint \frac{d\theta}{\sqrt{1 \pm k^2 \sin\theta}} = \frac{4}{\sqrt{1+k^2}} K\left(\sqrt{\frac{2k^2}{1+k^2}}\right)$$

$$\oint \frac{d\theta}{\sqrt{1+k^2 \pm 2k\cos\theta}} = 4K(k) \quad \text{Gradshteyn 3.674.1}$$

$$\oint \frac{d\theta}{\sqrt{1+k^2 \pm 2k\sin\theta}} = 4K(k) \quad \text{Gradshteyn 3.674.1}$$

$$\oint \frac{\cos\theta d\theta}{\sqrt{a - b\cos\theta}} = \frac{4\sqrt{a+b}}{b}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right]$$

$$\text{with } k = \sqrt{\frac{2b}{a+b}}$$

$$\oint \frac{\cos\theta d\theta}{\sqrt{1+k^2 - 2k\cos\theta}} = \frac{4}{k}\left(K(k) - E(k)\right) \quad \text{Gradshteyn 3.674.3}$$

The next batch is from Alexander Russell, 1907.

$$\int_0^{\pi/2} \frac{\sin^2\phi}{\sqrt{1 - k^2\sin^2\phi}} d\phi = \frac{1}{k^2}(K(k) - E(k))$$

$$\oint \frac{\sin^2\phi}{\sqrt{1 - k^2\sin^2\phi}} d\phi = \frac{4}{k^2}(K(k) - E(k))$$

$$\int_0^{\pi/2} \frac{\cos(2\phi)}{\sqrt{1 - k^2\sin^2\phi}} d\phi = \frac{2}{k^2}\left(E(k) - K(k)\right) + K(k)$$

$$\int_0^{\pi/2} \frac{\sin^2\phi\cos^2\phi}{\sqrt{1-k^2\sin^2\phi}}d\phi = \frac{2-k^2}{3k^4}E(k) - \frac{2-2k^2}{3k^4}K(k)$$

$$\int_0^{\pi/2} \frac{d\phi}{\left(1-k^2\sin^2\phi\right)^{3/2}} = \frac{E(k)}{1-k^2}$$

$$\int_0^{\pi/2} \frac{\sin^2\phi d\phi}{\left(1-k^2\sin^2\phi\right)^{3/2}} = \frac{E(k)}{k^2(1-k^2)} - \frac{K(k)}{k^2}$$

$$\int_0^{\pi/2} \sin^2\phi\sqrt{1-k^2\sin^2\phi}d\phi = \frac{3k^2-1}{3k^2}E(k) + \frac{1-k^2}{3k^2}K(k)$$

Derivative formulas with respect to k.

$$\frac{dE(k)}{dk} = \frac{E(k)-K(k)}{k}$$

$$\frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}$$

$$\frac{d^2E(k)}{dk^2} = -\frac{1}{k}\frac{dK(k)}{dk} = \frac{(1-k^2)K(k)-E(k)}{k^2(1-k^2)}$$

Change of parameter via Landen transformations. For these formulas, $k \geq 0$ and $w \geq 0$.

$$(1+k)K(k) = K\left(\frac{2\sqrt{k}}{1+k}\right)$$

$$K(k) = \frac{1}{1+k}K\left(\frac{2\sqrt{k}}{1+k}\right)$$

$$w = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}$$

$$k = \frac{2\sqrt{w}}{1+w}$$

$$K(k) = (1+w)K(w)$$

$$E(k) = \frac{2}{1+w}E(w) - (1-w)K(w)$$

The Landen transform w is much closer to zero than k. This is helpful in evaluating $K(k)$ as k approaches 1.

Some alternative definitions of $E(k)$ and $K(k)$ follow.

$$\begin{aligned} K(k) &= \frac{1}{4}\oint \frac{d\theta}{\sqrt{1+k^2-2k\cos\theta}} \\ E(k) &= \frac{1}{4}\oint \frac{(1-k\cos\theta)d\theta}{\sqrt{1+k^2-2k\cos\theta}} \end{aligned}$$

$$\oint \sqrt{1+k^2-2k\cos\theta}\,d\theta = 8E(k) + 4(k^2-1)K(k)$$

Routines to Calculate Complete Elliptic Integrals of First and Second Kind

Richard at `http://www.mymathlib.webtrellis.net/functions/elliptic_integrals.html` has provided beautiful examples of code to calculate elliptic integrals. I am very impressed.

Borrowing his interface, but using my implementation of the arithmetic-geometric mean provided in AMS-55, I offer the following totally free, open, code.

```
/*
    K_and_E.c is a little demo program in C demonstrating easy
    calculation of K and E.

    This program is totally freeware, enjoy.

    Compiling using GCC -
        gcc K_and_E.c -o K_and_E -lm
        ./E_and_E
*/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
```

```
int Complete_Elliptic_K_and_E
        (char arg, double parameter, double* K, double* E)
{
/*    Usage: the character arg is 'k', 'm', or 'a' to inform this
             routine of the type of parameter being used. Pointers
             are used for K and E so that we  can return multiple
     values without a struct.

       Example Call: err = Complete_Elliptic_K_and_E('k',k,&K,&E);

        License: Freeware by Kurt Nalty, 2011.

        Credits: Algorithm based upon
         Handbook of Mathematical Functions,
                Abramowitz and Stegun, National Bureau of Standards,
 pp 589-599

        Credit Coding Based After:

   http://mymathlib.webtrellis.net/functions/elliptic_integrals.html

                    (I very much liked his use of a letter parameter for
     function selection)

    returned code:  0 -> no problem
                       -1 -> bad parameter specification
                       -2 -> K sent to infinity by k=1, m=1, or
             alpha = 90 degrees

    Accuracy - seems to be within 5 digits easily.

*/

    double a[10],b[10],c[10];
    int i;
```

```
    double k, m, alpha;
    double pi = 3.141592653589793;
    double S, scale;
    double tol = 1.0e-10,err;    // initial tolerance

// Check arguments

    switch (arg) {
        case 'k':
            k = parameter;
            m = k*k;
            alpha = asin(k);
            break;
        case 'm':
            k = sqrt(parameter);
            m = parameter;
            alpha = asin(k);
            break;
        case 'a':
            k = sin(parameter);
            m = k*k;
            alpha = parameter;
            break;
                default:
            *K = 1.0E30;    // infinity approximated
            *E = 1.0;
            return (-2);    // bad argument
        }


/*
    K(m)=/int^{\pi/2}_{0} \left(1-m \sin^2 \theta \right)^{-1/2} d \theta
    E(m)=/int^{\pi/2}_{0} \left(1-m \sin^2 \theta \right)^{1/2}  d \theta

    For the AGM process

    a[0] = 1.0     b[0] = cos(alpha) = sqrt(1-k*k)
                             c[0] = sin(alpha) = k
```

```
    a[i]=0.5*(a[i-1]+b[i-1])  b[i]=sqrt(a[i-1]*b[i-1])
                             c[i]=0.5*(a[i-1]-b[i-1])

    when c[i] is small enough,

    K(alpha) = pi/(2 a[n])

    S = 0.5*Sigma_i (2^i c_i^2) = Sigma_i (2^(i-1) c_i^2)
*/

    a[0] = 1.0;  b[0] = cos(alpha);   c[0] = sin(alpha);
    S = 0.5*c[0]*c[0];  scale = 1.0;

    for (i=1;i<9;i++) {  // max interation depth is 10.
                         // Break if within tolerance

        a[i] = 0.5*(a[i-1] + b[i-1]);
        b[i] = sqrt(a[i-1]*b[i-1]);
        c[i] = 0.5*(a[i-1] - b[i-1]);
        S += scale*c[i]*c[i];
        scale *= 2.0;
        err = (c[i]);
        if (err < 0.0) err *= -1.0;    // take magnitude
        if(err <= tol) break;

    }
    *K = pi/(2.0*a[i]);
    *E = *K - *K*S;
    if (i==9) return(-2);    // no convergence

    return 0;

}


int main(void)
{
```

```
    int i;
    double k, E, K;

    FILE* EFile;
    FILE* KFile;


// We provide a table of value for E(k) and K(k)
// for comparison against references.

    EFile = fopen("E(k)","w");
    KFile = fopen("K(k)","w");
    fprintf(EFile,"k         E(k)  \n\n");
    fprintf(KFile,"k         K(k)  \n\n");
    for (i=0;i<100;i++) {
        k = i/100.0;
        Complete_Elliptic_K_and_E('k',k,&K, &E);
        fprintf(EFile,"%g   %g  \n",k,E);
        fprintf(KFile,"%g   %g  \n",k,K);
    }
    fclose(EFile);
    fclose(KFile);


}
```

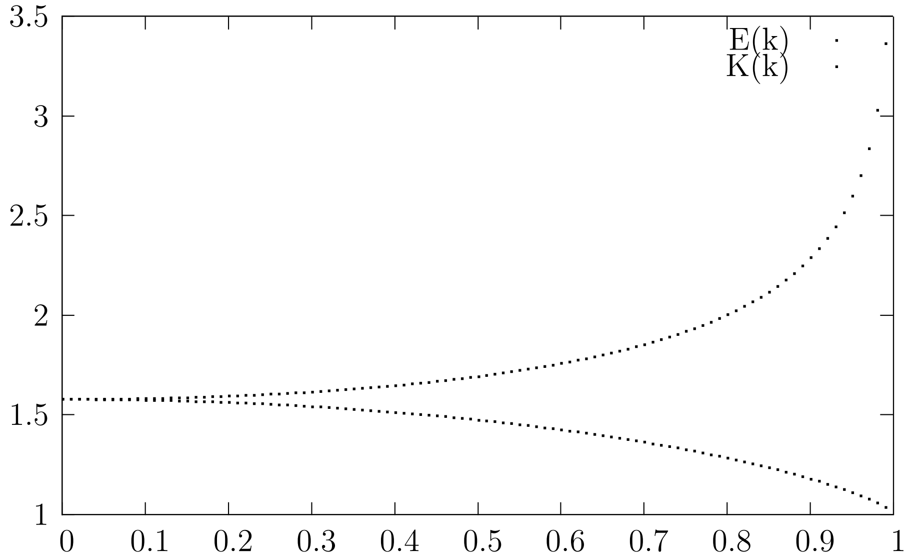
Using the output files E(k) and K(k), trimming the headers, and plotting these with GnuPlot, we have the following figure.

Derivations Useful for Solenoids and Coils

When making magnetic fields calculations, we naturally encounter complete elliptic integrals of the first and second kind, using a form not commonly tabulated in tables of integrals.

We achieve the useful form by substituting a double angle form for the $\sin\theta$ terms in the defining equations. In the sequence below, I introduce yet another variation

Plot of K(k) and E(k)



in the definition for $E(K)$, using the angular definition with parameter k. This is simply the result of substituting $t = \sin\theta$ in the first equation.

$$
\begin{aligned}
E(k) &= \int_0^1 \sqrt{\frac{(1-k^2t^2)}{(1-t^2)}}dt \\
&= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)}d\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\theta=\pi/2} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)}d(2\theta) \\
&= \frac{1}{4}\int_{-\pi}^{\phi=\pi} \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi \\
&= \frac{1}{4}\oint \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi
\end{aligned}
$$

Now, we are going to clean up and re-arrange some terms. The range for k is $0 \leq k^2 \leq 1$. We will introduce $b^2 = k^2/(2-k^2)$ shortly which will also have the range $0 \leq b^2 \leq 1$.

$$
\begin{aligned}
E(k) &= \frac{1}{4}\oint \sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}d\phi \\
&= \frac{1}{4}\oint \sqrt{\left(\frac{2-k^2}{2}\right)-\frac{k^2}{2}\cos\phi d\phi} \\
&= \frac{1}{4}\sqrt{\frac{2-k^2}{2}}\oint \sqrt{1-\left(\frac{k^2}{2-k^2}\right)\cos\phi d\phi}
\end{aligned}
$$

We now introduce b^2, and re-arrange some more.

$$
\begin{aligned}
E(k) &= \frac{1}{4}\sqrt{\frac{1}{1+b^2}}\oint\sqrt{1-b^2\cos\phi}\,d\phi \\
\oint\sqrt{1-b^2\cos\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\text{with } k &= \sqrt{\frac{2b^2}{1+b^2}}
\end{aligned}
$$

Now this formula is very useful for circular current loops. Now, since we are doing a full circular integration, we can start anywhere we like, as far as an arbitrary phase shift in the angular variable. Using shifts of ± 90 deg and 180 deg, we get the related set of equivalents.

$$
\begin{aligned}
\oint\sqrt{1-b^2\cos\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\oint\sqrt{1+b^2\cos\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\oint\sqrt{1-b^2\sin\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\oint\sqrt{1+b^2\sin\phi}\,d\phi &= 4\sqrt{1+b^2}E(k) \\
\text{with } k &= \sqrt{\frac{2b^2}{1+b^2}}
\end{aligned}
$$

We now do the similar transformations for the complete elliptical integral of the first kind.

$$
\begin{aligned}
K(k) = F(1,k) &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\
&= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}
\end{aligned}
$$

We now double our angle, as before.

$$
\begin{aligned}
K(k) &= \frac{1}{4}\int_{-\pi/2}^{\theta=\pi/2} \frac{d(2\theta)}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\left(2\theta\right)\right)}} \\
&= \frac{1}{4}\int_{-\pi}^{\pi=\pi} \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}} \\
&= \frac{1}{4}\oint \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}}
\end{aligned}
$$

We bring out a common factor b^2 as before.

$$
\begin{aligned}
K(k) &= \frac{1}{4}\oint \frac{d\phi}{\sqrt{1-k^2\left(\frac{1}{2}-\frac{1}{2}\cos\phi\right)}} \\
&= \frac{1}{4}\oint \frac{d\phi}{\sqrt{\frac{2-k^2}{2}-\frac{k^2}{2}\cos\phi}} \\
&= \frac{1}{4}\sqrt{\frac{2}{2-k^2}}\oint \frac{d\phi}{\sqrt{1-\frac{k^2}{2-k^2}\cos\phi}} \\
&= \frac{1}{4}\sqrt{1+b^2}\oint \frac{d\phi}{\sqrt{1-b^2\cos\phi}}
\end{aligned}
$$

We now find the four similar results

$$
\begin{aligned}
\oint \frac{d\phi}{\sqrt{1-b^2\cos\phi}} &= \frac{4}{\sqrt{1+b^2}}K(k) \\
\oint \frac{d\phi}{\sqrt{1+b^2\cos\phi}} &= \frac{4}{\sqrt{1+b^2}}K(k) \\
\oint \frac{d\phi}{\sqrt{1-b^2\sin\phi}} &= \frac{4}{\sqrt{1+b^2}}K(k) \\
\oint \frac{d\phi}{\sqrt{1+b^2\sin\phi}} &= \frac{4}{\sqrt{1+b^2}}K(k) \\
\text{with}\quad k &= \sqrt{\frac{2b^2}{1+b^2}}
\end{aligned}
$$

Derivatives with Respect to Parameters

Due to the variation in definitions of the elliptic integrals, it is easy to make mistakes with the derivatives as well. Always be sure of the definitions used in your reference.

For these formulas, I am using the k style formulas.

I'll start with the easiest one.

$$\begin{aligned}
E(k,\phi) &= \int_0^\phi \sqrt{1-k^2\sin^2\theta}d\theta \\
\frac{\partial E(k,\phi)}{\partial k} &= \int_0^\phi \frac{\partial}{\partial k}\sqrt{1-k^2\sin^2\theta}d\theta \\
&= \frac{1}{2}\int_0^\phi \left(1-k^2\sin^2\theta\right)^{-1/2}\left(-2k\sin^2\theta\right)d\theta \\
&= \frac{1}{k}\int_0^\phi \frac{\left(-k^2\sin^2\theta\right)d\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= \frac{1}{k}\int_0^\phi \frac{\left[-1+\left(1-k^2\sin^2\theta\right)\right]d\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= -\frac{1}{k}\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \frac{1}{k}\int_0^\phi \sqrt{1-k^2\sin^2\theta}d\theta \\
&= -\frac{F(k,\phi)+E(k,\phi)}{k} \\
\frac{\partial E(k,\phi)}{\partial k} &= \frac{E(k,\phi)-F(k,\phi)}{k}
\end{aligned}$$

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