Curves of Constant Curvatures in Four Dimensional Spacetime

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Abstract

Curves of constant curvature κ, torsion τ and lift γ trace curves on the surface of a hypersphere with fixed radius $R^2 = (\tau^2 + \gamma^2)/(\kappa^2\gamma^2)$. This can be further decomposed into two simple circular orbits of radius r_1 with spatial frequence ω_1 and r_2 with spatial frequency ω_2, in mutually orthogonal planes.

Curvatures

On a line, distance s measures the deviation from a fixed point. In the plane, a circle is a path of constant curvature, where the curvature (κ) measures deviation from a line. In space, a helix is a path of constant curvature and torsion, where torsion (τ) measures the deviation out of the plane, while curvature measures the deviation from a line. As we move to four dimensional spacetime, we encounter a new curvature, lift (γ), which measures the deviation from a volume.

In the odd numbered dimension of a line, distance and coordinate span the entire universe from -∞ to +∞, but in the even numbered dimension space of a plane, while the distance does span infinity, coordinates on the curve are bound to a finite range of $\pm r = \pm 1/\kappa$. In other words, the circle is localized. When we are in the third dimension, the effect of torsion is to allow the axis of the helix to once again span the entire space, while the radial component remains bound. Once we go to the even dimensions of spacetime, we find that the curves of constant curvature, torsion and lift have bound coordinates limited to $\pm\sqrt{(\tau^2 + \gamma^2)/(\kappa^2\gamma^2)}$.

Frenet-Serret Curve Specification

Frenet and Serret parameterized curves along their pathlength s by the curvature κ and torsion τ. Their formula can be extended to higher dimensions. In this case, I will use a left hand coordinate system in four dimensions, and characterize the curves by curvature κ, torsion τ and lift γ. Analogous to arrows over symbols indicating threevectors, a tilde over a symbol in this paper indicates a fourvector. For example, $\tilde{r} = (r_x, r_y, r_z, r_t)$. Using this notation, the Frenet-Serret equations, using a left hand four dimensional coordinate system, parameterized by pathlength s, are

$$\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} \\
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\frac{d\tilde{w}}{ds} &= -\gamma\tilde{b}
\end{aligned}$$

These equations are easily solved numerically by Runge-Kutta and other algorithms. The demonstration program used for these illustrations is RK4.c, available at `http://www.kurtnalty.com/RK4.c`. A solution for $\kappa = \tau = \gamma = 1$ is shown at the top of the next page.

Chain of Substitutions

Assuming constant curvatures, so that their derivatives are zero, we have a chain of substitutions leading the differential equation for $\tilde{u}$.

$$\begin{aligned}
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} \\
\tilde{n} &= \frac{1}{\kappa}\frac{d\tilde{u}}{ds} \\
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u}
\end{aligned}$$

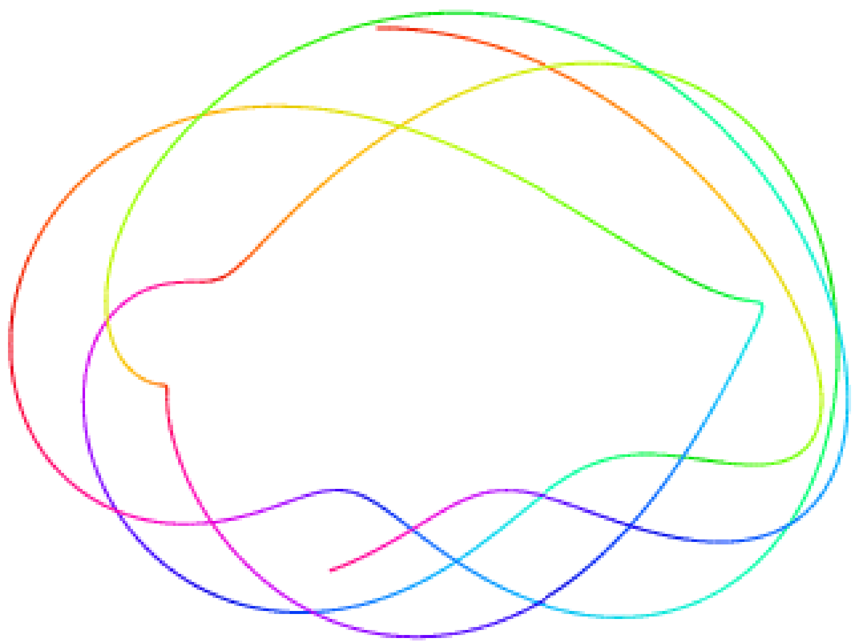

Figure 1: $\kappa = \tau = \gamma = 1$ Curve (1000 points)

$$
\begin{aligned}
\tilde{b} &= \frac{1}{\tau}\frac{d\tilde{n}}{ds} + \frac{\kappa}{\tau}\tilde{u} \\
\tilde{b} &= \frac{1}{\kappa\tau}\frac{d^2\tilde{u}}{ds^2} + \frac{\kappa}{\tau}\tilde{u} \\
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\tilde{w} &= \frac{1}{\gamma}\frac{d\tilde{b}}{ds} + \frac{\tau}{\gamma}\tilde{n} \\
\tilde{w} &= \frac{1}{\kappa\tau\gamma}\frac{d^3\tilde{u}}{ds^3} + \frac{\kappa}{\tau\gamma}\frac{d\tilde{u}}{ds} + \frac{\tau}{\kappa\gamma}\frac{d\tilde{u}}{ds} \\
\frac{d\tilde{w}}{ds} &= -\gamma\tilde{b} \\
\frac{1}{\kappa\tau\gamma}\frac{d^4\tilde{u}}{ds^4} + \frac{\kappa}{\tau\gamma}\frac{d^2\tilde{u}}{ds^2} + \frac{\tau}{\kappa\gamma}\frac{d^2\tilde{u}}{ds^2} &= -\frac{\gamma}{\kappa\tau}\frac{d^2\tilde{u}}{ds^2} - \frac{\kappa\gamma}{\tau}\tilde{u}
\end{aligned}
$$

Cleaning up, we have the defining partial differential equation for $\tilde{u}$

$$\frac{d^4\tilde{u}}{ds^4} + \left(\kappa^2 + \tau^2 + \gamma^2\right) \frac{d^2\tilde{u}}{ds^2} + \kappa^2\gamma^2\tilde{u} = 0$$

Expecting periodic solutions, we take the Laplace transform of the $\tilde{u}$ equations. In the Laplace transform equations, S will be the complex frequency rather than the path length above.

$$\begin{aligned}
\frac{d^4\tilde{u}}{ds^4} + \left(\kappa^2 + \tau^2 + \gamma^2\right) \frac{d^2\tilde{u}}{ds^2} + \kappa^2\gamma^2\tilde{u} &= 0 \\
S^4 + \left(\kappa^2 + \tau^2 + \gamma^2\right) S^2 + \kappa^2\gamma^2 &= 0
\end{aligned}$$

This leads to

$$S^2 = \frac{-\left(\kappa^2 + \tau^2 + \gamma^2\right) \pm \sqrt{\left(\kappa^2 + \tau^2 + \gamma^2\right)^2 - 4\kappa^2\gamma^2}}{2} \leq 0$$

Consequently, for each component of the unit tangent, we have expressions of the form

$$u_t = A\cos(\omega_1 s + \phi) + B\cos(\omega_2 s + \theta)$$

where the A, B, ϕ and θ are set by initial conditions, and

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{\left(\kappa^2 + \tau^2 + \gamma^2\right) + \sqrt{\left(\kappa^2 + \tau^2 + \gamma^2\right)^2 - 4\kappa^2\gamma^2}}{2}} \\
\omega_2 &= \sqrt{\frac{\left(\kappa^2 + \tau^2 + \gamma^2\right) - \sqrt{\left(\kappa^2 + \tau^2 + \gamma^2\right)^2 - 4\kappa^2\gamma^2}}{2}}
\end{aligned}$$

Some very useful relationships exist between these two frequencies.

From the defining differential equation, we have (keeping the positive roots)

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= \kappa^2 + \tau^2 + \gamma^2 \\
\omega_1^2 * \omega_2^2 &= \kappa^2\gamma^2 \\
\omega_1 * \omega_2 &= \kappa\gamma \\
\frac{\omega_1}{\omega_2} &= \frac{\kappa\gamma}{\omega_2^2} = \frac{\omega_1^2}{\kappa\gamma}
\end{aligned}$$

Harmonic Solutions Yield Closed Strings

The two frequencies in the previous solutions are usually aharmonic. The trajectories traced by the particle cover a surface, rather than forming a filament. This is illustrated in Figure 2 on the next page, showing 3000 points of the 111 solution with axis, slightly rotated.

To drop from surface coverage to closed string status requires a harmonic connection between the two frequencies. This is shown in the 3 4 5 solution, above. Notice the filamentary solution.

The ratio of the two frequencies is

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{(\kappa^2+\tau^2+\gamma^2)-\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}}$$

We see that κ and γ are interchangeable in the above formula. When κ, τ and γ form a Pythagorean triple, we get harmonic solutions.

$$\begin{aligned}
\kappa^2+\tau^2 &= \gamma^2 \\
\frac{\omega_1}{\omega_2} &= \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{(\kappa^2+\tau^2+\gamma^2)-\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}} \\
&= \sqrt{\frac{(2\gamma^2)+\sqrt{(2\gamma^2)^2-4\kappa^2\gamma^2}}{(2\gamma^2)-\sqrt{(2\gamma^2)^2-4\kappa^2\gamma^2}}} \\
&= \sqrt{\frac{\gamma+\sqrt{\gamma^2-\kappa^2}}{\gamma-\sqrt{\gamma^2-\kappa^2}}} \\
&= \sqrt{\frac{\gamma+\tau}{\gamma-\tau}} = \sqrt{\frac{(\gamma+\tau)*(\gamma+\tau)}{(\gamma-\tau)*(\gamma+\tau)}} \\
&= \sqrt{\frac{(\gamma+\tau)^2}{\gamma^2-\tau^2}} = \sqrt{\frac{(\gamma+\tau)^2}{\kappa^2}} \\
\frac{\omega_1}{\omega_2} &= \frac{\gamma+\tau}{\kappa}
\end{aligned}$$

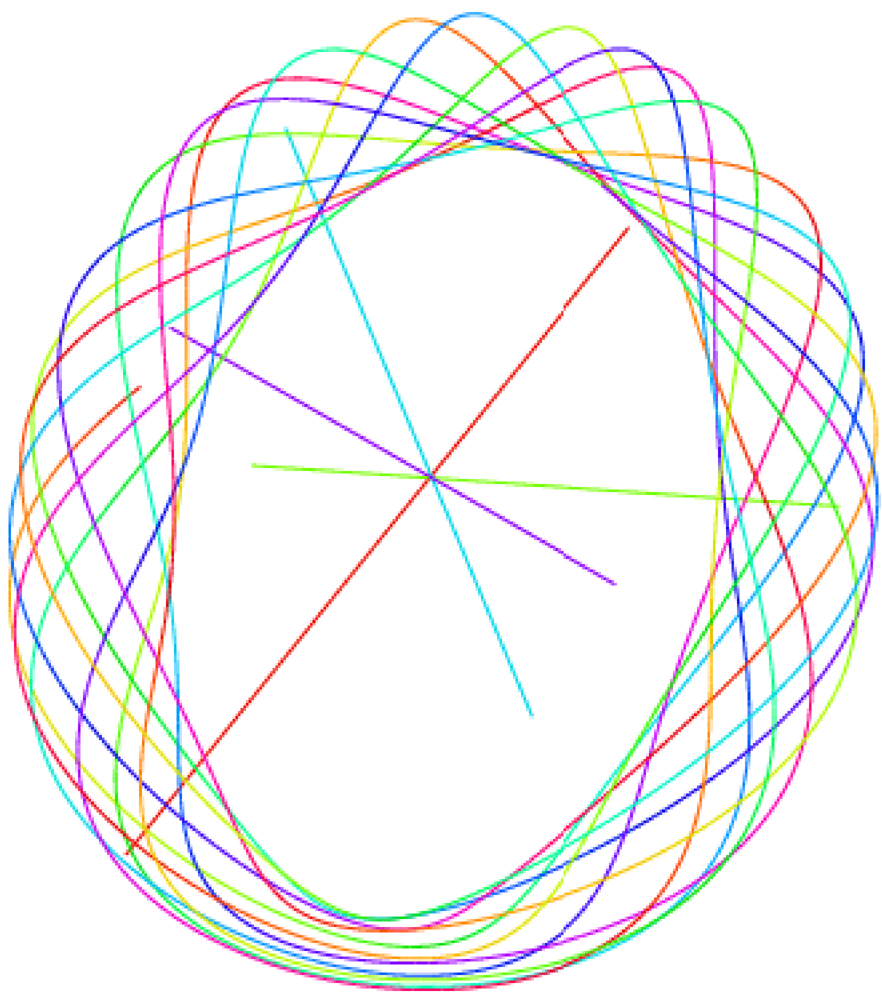

Figure 2: 3000 Points of 111 Solution

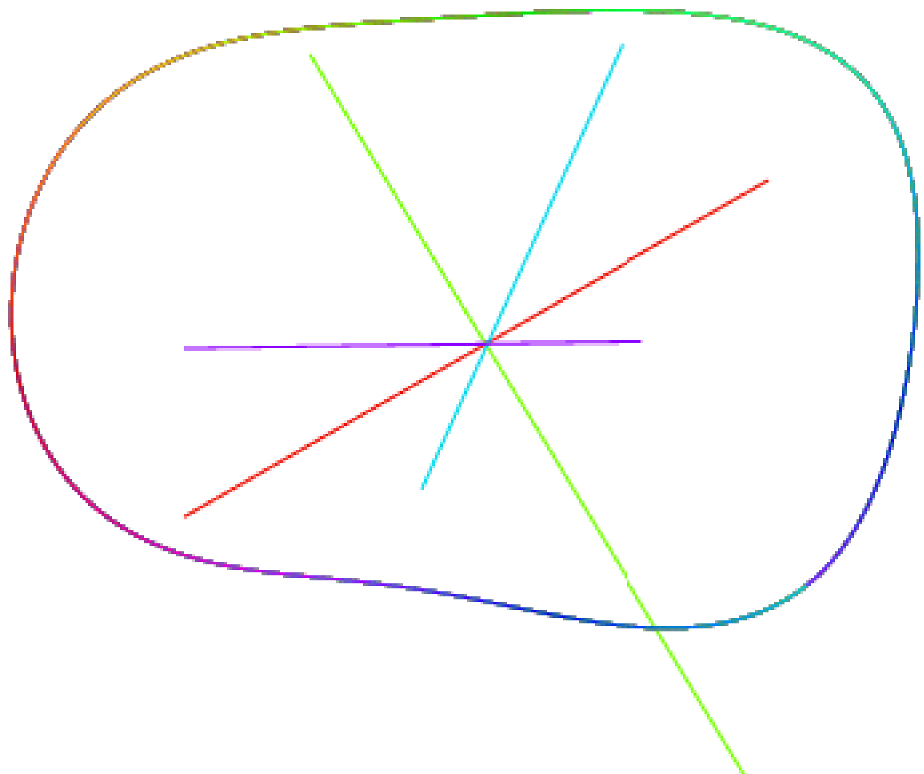

Figure 3: 3000 Points of 345 Solution

As an example, using the common 3,4,5 triangle as κ, τ, and γ, we have

$$\frac{\omega_1}{\omega_2} = \frac{\gamma+\tau}{\kappa} = \frac{9}{3} = 3$$

Likewise, interchanging κ and τ, we have 4,3,5 and

$$\frac{\omega_1}{\omega_2} = \frac{\gamma+\tau}{\kappa} = \frac{8}{4} = 2$$

Using larger triples, we can get fractional ratios. For example, 48,55,73 yields 8/3 for the ratio.

In all these harmonically related cases, the curve is periodic and traces over itself, forming a filament as opposed to covering a surface.

More General Harmonic Case

In the previous section, the curvatures were chosen to be a Pythagorean triple, yet the simple triad 10,11,12 also yields a harmonic solution with ratio 8/3, as shown in Figure 4 on the next page. We now find a more general case for harmonic solutions by simplifying the expression under the radical in the formula for the harmonic ratio.

Begin again with

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{(\kappa^2+\tau^2+\gamma^2)-\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}}$$

Multiply top and bottom under the outer radical with the numerator. We get

$$\begin{aligned}\frac{\omega_1}{\omega_2} &= \frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-(\kappa^2+\tau^2+\gamma^2)^2+4\kappa^2\gamma^2}} \\ &= \frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{\sqrt{4\kappa^2\gamma^2}} \\ &= \frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{(\kappa^2+\tau^2+\gamma^2)^2-4\kappa^2\gamma^2}}{2\kappa\gamma}\end{aligned}$$

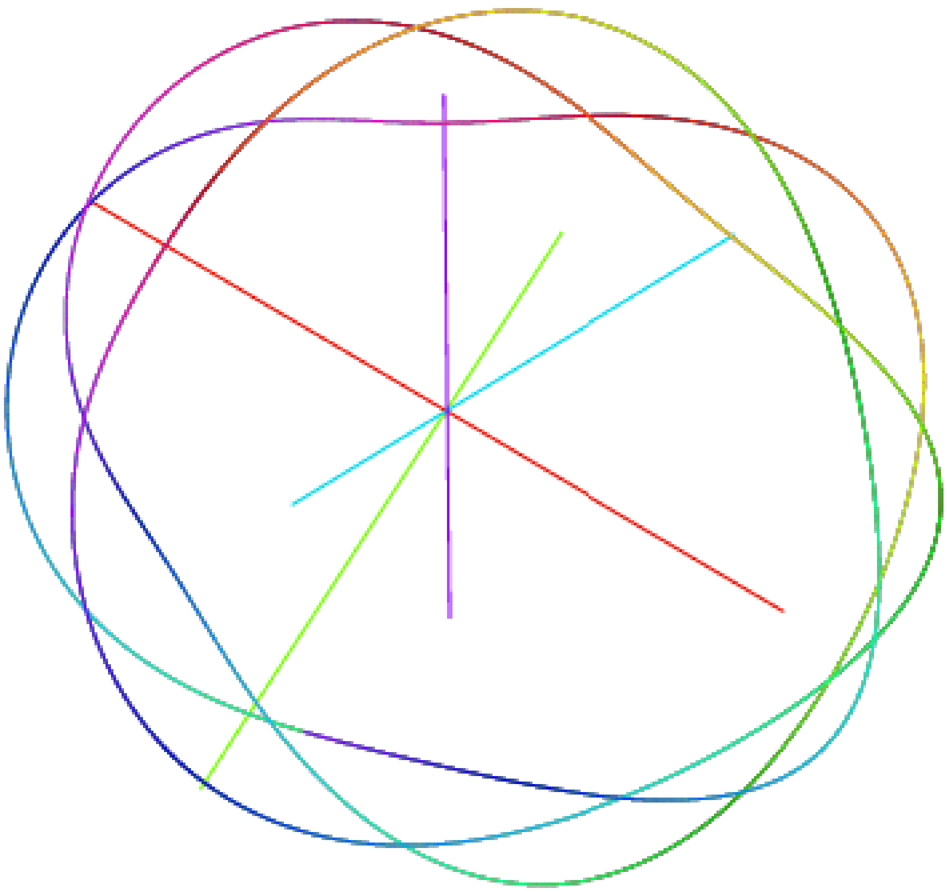

Figure 4: 10 11 12 Solution

We find a more relaxed condition for harmonic solution to be that, for integral values for κ, τ and γ, that $\sqrt{(\kappa^2+\tau^2+\gamma^2)^2 - 4\kappa^2\gamma^2}$ be an integer, or that $(\kappa^2+\tau^2+\gamma^2)$ be the hypotenuse of a Pythagorean triple, with $2\kappa\gamma$ being one of the sides.

Pythagorean Triples

Pythagorean triples are integers which satisfy $a^2+b^2=c^2$. To generate these triples, we can first decide how much difference should be between the second and third numbers.

$$\begin{aligned} a^2+b^2 &= (b+d)^2 \\ a^2+b^2 &= b^2+2bd+d^2 \\ a^2-d^2 &= 2bd \\ b &= \frac{a^2-d^2}{2d} \end{aligned}$$

Given a value a and a difference d, we have a triple

$$a, \frac{a^2-d^2}{2d}, \frac{a^2+d^2}{2d}$$

Constant Curvatures

We now solve the differential equations letting κ, τ and γ be independent constants.

The components of $\tilde{u}$ are

$$\begin{aligned} u_x(s) &= A_x\cos(\omega_1 s+\phi_x) + B_x\cos(\omega_2 s+\theta_x) \\ u_y(s) &= A_y\cos(\omega_1 s+\phi_y) + B_y\cos(\omega_2 s+\theta_y) \\ u_z(s) &= A_z\cos(\omega_1 s+\phi_z) + B_z\cos(\omega_2 s+\theta_z) \\ u_t(s) &= A_t\cos(\omega_1 s+\phi_t) + B_y\cos(\omega_2 s+\theta_t) \end{aligned}$$

For $\tilde{n}$, we have

$$\begin{aligned}
\tilde{n} &= \frac{1}{\kappa}\frac{d\tilde{u}}{ds} \\
n_x(s) &= \left(\frac{-\omega_1}{\kappa}\right) A_x \sin(\omega_1 s + \phi_x) + \left(\frac{-\omega_2}{\kappa}\right) B_x \sin(\omega_2 s + \theta_x) \\
n_y(s) &= \left(\frac{-\omega_1}{\kappa}\right) A_y \sin(\omega_1 s + \phi_y) + \left(\frac{-\omega_2}{\kappa}\right) B_y \sin(\omega_2 s + \theta_y) \\
n_z(s) &= \left(\frac{-\omega_1}{\kappa}\right) A_z \sin(\omega_1 s + \phi_z) + \left(\frac{-\omega_2}{\kappa}\right) B_z \sin(\omega_2 s + \theta_z) \\
n_t(s) &= \left(\frac{-\omega_1}{\kappa}\right) A_t \sin(\omega_1 s + \phi_t) + \left(\frac{-\omega_2}{\kappa}\right) B_t \sin(\omega_2 s + \theta_t)
\end{aligned}$$

For $\tilde{b}$, we have

$$\begin{aligned}
\tilde{b} &= \frac{1}{\tau}\frac{d\tilde{n}}{ds} + \frac{\kappa}{\tau}\tilde{u} \\
b_x(s) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_x \cos(\omega_1 s + \phi_x) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_x \cos(\omega_2 s + \theta_x) \\
b_y(s) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_y \cos(\omega_1 s + \phi_y) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_y \cos(\omega_2 s + \theta_y) \\
b_z(s) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_z \cos(\omega_1 s + \phi_z) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_z \cos(\omega_2 s + \theta_z) \\
b_t(s) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_t \cos(\omega_1 s + \phi_t) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_t \cos(\omega_2 s + \theta_t)
\end{aligned}$$

For $\tilde{w}$, we have

$$
\begin{aligned}
\tilde{w} &= \frac{1}{\gamma}\frac{d\tilde{b}}{ds} + \frac{\tau}{\gamma}\tilde{n} \\
w_x(s) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_x \sin(\omega_1 s + \phi_x) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_x \sin(\omega_2 s + \theta_x) \\
w_y(s) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_y \sin(\omega_1 s + \phi_y) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_y \sin(\omega_2 s + \theta_y) \\
w_z(s) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_z \sin(\omega_1 s + \phi_z) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_z \sin(\omega_2 s + \theta_z) \\
w_t(s) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_t \sin(\omega_1 s + \phi_t) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_t \sin(\omega_2 s + \theta_t)
\end{aligned}
$$

The $\tilde{u}$ and $\tilde{b}$ initial conditions allow us to find the magnitude/cosine terms.

$$
\begin{aligned}
u_x(0) &= A_x \cos(\phi_x) + B_x \cos(\theta_x) \\
u_y(0) &= A_y \cos(\phi_y) + B_y \cos(\theta_y) \\
u_z(0) &= A_z \cos(\phi_z) + B_z \cos(\theta_z) \\
u_t(0) &= A_t \cos(\phi_t) + B_y \cos(\theta_t) \\
b_x(0) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_x \cos(\phi_x) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_x \cos(\theta_x) \\
b_y(0) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_y \cos(\phi_y) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_y \cos(\theta_y) \\
b_z(0) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_z \cos(\phi_z) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_z \cos(\theta_z) \\
b_t(0) &= \left(\frac{\kappa}{\tau} - \frac{\omega_1^2}{\kappa\tau}\right) A_t \cos(\phi_t) + \left(\frac{\kappa}{\tau} - \frac{\omega_2^2}{\kappa\tau}\right) B_t \cos(\theta_t)
\end{aligned}
$$

Solving these pairs of equations, we get

$$
\begin{aligned}
A_x \cos(\phi_x) &= \frac{(\kappa^2 - \omega_2^2)\, u_x(0) - \kappa\tau b_x(0)}{\omega_1^2 - \omega_2^2} \\
A_y \cos(\phi_y) &= \frac{(\kappa^2 - \omega_2^2)\, u_y(0) - \kappa\tau b_y(0)}{\omega_1^2 - \omega_2^2} \\
A_z \cos(\phi_z) &= \frac{(\kappa^2 - \omega_2^2)\, u_z(0) - \kappa\tau b_z(0)}{\omega_1^2 - \omega_2^2} \\
A_t \cos(\phi_t) &= \frac{(\kappa^2 - \omega_2^2)\, u_t(0) - \kappa\tau b_t(0)}{\omega_1^2 - \omega_2^2} \\
B_x \cos(\theta_x) &= \frac{-(\kappa^2 - \omega_1^2)\, u_x(0) + \kappa\tau b_x(0)}{\omega_1^2 - \omega_2^2} \\
B_y \cos(\theta_y) &= \frac{-(\kappa^2 - \omega_1^2)\, u_y(0) + \kappa\tau b_y(0)}{\omega_1^2 - \omega_2^2} \\
B_z \cos(\theta_z) &= \frac{-(\kappa^2 - \omega_1^2)\, u_z(0) + \kappa\tau b_z(0)}{\omega_1^2 - \omega_2^2} \\
B_t \cos(\theta_t) &= \frac{-(\kappa^2 - \omega_1^2)\, u_t(0) + \kappa\tau b_t(0)}{\omega_1^2 - \omega_2^2}
\end{aligned}
$$

The $\tilde{n}$ and $\tilde{w}$ initial conditions allow us to find the magnitude/sine terms.

$$
\begin{aligned}
n_x(0) &= \left(\frac{-\omega_1}{\kappa}\right) A_x \sin(\phi_x) + \left(\frac{-\omega_2}{\kappa}\right) B_x \sin(\theta_x) \\
n_y(0) &= \left(\frac{-\omega_1}{\kappa}\right) A_y \sin(\phi_y) + \left(\frac{-\omega_2}{\kappa}\right) B_y \sin(\theta_y) \\
n_z(0) &= \left(\frac{-\omega_1}{\kappa}\right) A_z \sin(\phi_z) + \left(\frac{-\omega_2}{\kappa}\right) B_z \sin(\theta_z) \\
n_t(0) &= \left(\frac{-\omega_1}{\kappa}\right) A_t \sin(\phi_t) + \left(\frac{-\omega_2}{\kappa}\right) B_t \sin(\theta_t) \\
w_x(0) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_x \sin(\phi_x) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_x \sin(\theta_x) \\
w_y(0) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_y \sin(\phi_y) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_y \sin(\theta_y) \\
w_z(0) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_z \sin(\phi_z) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_z \sin(\theta_z) \\
w_t(0) &= \left(\frac{\omega_1^3}{\kappa\tau\gamma} - \omega_1\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) A_t \sin(\phi_t) \\
&\quad + \left(\frac{\omega_2^3}{\kappa\tau\gamma} - \omega_2\left(\frac{\kappa}{\tau\gamma} + \frac{\tau}{\kappa\gamma}\right)\right) B_t \sin(\theta_t)
\end{aligned}
$$

Solving these, we find

$$
\begin{aligned}
A_x \sin(\phi_x) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2 - (\kappa^2+\tau^2))\, n_x(0) + \tau\gamma w_x(0)}{\omega_1^2 - \omega_2^2}\right] \\
A_y \sin(\phi_y) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2 - (\kappa^2+\tau^2))\, n_y(0) + \tau\gamma w_y(0)}{\omega_1^2 - \omega_2^2}\right] \\
A_z \sin(\phi_z) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2 - (\kappa^2+\tau^2))\, n_z(0) + \tau\gamma w_z(0)}{\omega_1^2 - \omega_2^2}\right] \\
A_t \sin(\phi_t) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2 - (\kappa^2+\tau^2))\, n_t(0) + \tau\gamma w_t(0)}{\omega_1^2 - \omega_2^2}\right] \\
B_x \sin(\theta_x) &= -\frac{\kappa}{\omega_2}\left[\frac{(\omega_1^2 - (\kappa^2+\tau^2))\, n_x(0) + \tau\gamma w_x(0)}{\omega_1^2 - \omega_2^2}\right] \\
B_y \sin(\theta_y) &= -\frac{\kappa}{\omega_2}\left[\frac{(\omega_1^2 - (\kappa^2+\tau^2))\, n_y(0) + \tau\gamma w_y(0)}{\omega_1^2 - \omega_2^2}\right] \\
B_z \sin(\theta_z) &= -\frac{\kappa}{\omega_2}\left[\frac{(\omega_1^2 - (\kappa^2+\tau^2))\, n_z(0) + \tau\gamma w_z(0)}{\omega_1^2 - \omega_2^2}\right] \\
B_t \sin(\theta_t) &= -\frac{\kappa}{\omega_2}\left[\frac{(\omega_1^2 - (\kappa^2+\tau^2))\, n_t(0) + \tau\gamma w_t(0)}{\omega_1^2 - \omega_2^2}\right]
\end{aligned}
$$

Solutions have Constant Radius

Numeric evaluation shows constant radius solutions. We now find the same result from the algebraic representations above.

Begin with the unit tangent generic formula

$$
\begin{aligned}
u_x(s) &= A_x \cos(\omega_1 s + \phi_x) + B_x \cos(\omega_2 s + \theta_x) \\
u_y(s) &= A_y \cos(\omega_1 s + \phi_y) + B_y \cos(\omega_2 s + \theta_y) \\
u_z(s) &= A_z \cos(\omega_1 s + \phi_z) + B_z \cos(\omega_2 s + \theta_z) \\
u_t(s) &= A_t \cos(\omega_1 s + \phi_t) + B_y \cos(\omega_2 s + \theta_t)
\end{aligned}
$$

Integrate for position, arbitrarily keeping constants of integration zero.

$$\begin{aligned}
r_x(s) &= \frac{A_x}{\omega_1}\sin(\omega_1 s + \phi_x) + \frac{B_x}{\omega_2}\sin(\omega_2 s + \theta_x) \\
r_y(s) &= \frac{A_y}{\omega_1}\sin(\omega_1 s + \phi_y) + \frac{B_y}{\omega_2}\sin(\omega_2 s + \theta_y) \\
r_z(s) &= \frac{A_z}{\omega_1}\sin(\omega_1 s + \phi_z) + \frac{B_z}{\omega_2}\sin(\omega_2 s + \theta_z) \\
r_t(s) &= \frac{A_t}{\omega_1}\sin(\omega_1 s + \phi_t) + \frac{B_t}{\omega_2}\sin(\omega_2 s + \theta_y)
\end{aligned}$$

Choose initial conditions

$$\begin{aligned}
\tilde{u}(0) &= (1, 0, 0, 0) \\
\tilde{n}(0) &= (0, -1, 0, 0) \\
\tilde{b}(0) &= (0, 0, 1, 0) \\
\tilde{w}(0) &= (0, 0, 0, -1)
\end{aligned}$$

We now solve for the A, B, and angles

$$
\begin{aligned}
A_x \cos(\phi_x) &= \frac{(\kappa^2-\omega_2^2)\,u_x(0)-\kappa\tau b_x(0)}{\omega_1^2-\omega_2^2} = \frac{(\kappa^2-\omega_2^2)}{\omega_1^2-\omega_2^2} \\
A_y \cos(\phi_y) &= \frac{(\kappa^2-\omega_2^2)\,u_y(0)-\kappa\tau b_y(0)}{\omega_1^2-\omega_2^2} = 0 \\
A_z \cos(\phi_z) &= \frac{(\kappa^2-\omega_2^2)\,u_z(0)-\kappa\tau b_z(0)}{\omega_1^2-\omega_2^2} = \frac{-\kappa\tau}{\omega_1^2-\omega_2^2} \\
A_t \cos(\phi_t) &= \frac{(\kappa^2-\omega_2^2)\,u_t(0)-\kappa\tau b_t(0)}{\omega_1^2-\omega_2^2} = 0 \\
B_x \cos(\theta_x) &= \frac{-(\kappa^2-\omega_1^2)\,u_x(0)+\kappa\tau b_x(0)}{\omega_1^2-\omega_2^2} = \frac{-(\kappa^2-\omega_1^2)}{\omega_1^2-\omega_2^2} \\
B_y \cos(\theta_y) &= \frac{-(\kappa^2-\omega_1^2)\,u_y(0)+\kappa\tau b_y(0)}{\omega_1^2-\omega_2^2} = 0 \\
B_z \cos(\theta_z) &= \frac{-(\kappa^2-\omega_1^2)\,u_z(0)+\kappa\tau b_z(0)}{\omega_1^2-\omega_2^2} = \frac{\kappa\tau}{\omega_1^2-\omega_2^2} \\
B_t \cos(\theta_t) &= \frac{-(\kappa^2-\omega_1^2)\,u_t(0)+\kappa\tau b_t(0)}{\omega_1^2-\omega_2^2} = 0 \\
A_x \sin(\phi_x) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))\,n_x(0)+\tau\gamma w_x(0)}{\omega_1^2-\omega_2^2}\right] = 0 \\
A_y \sin(\phi_y) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))\,n_y(0)+\tau\gamma w_y(0)}{\omega_1^2-\omega_2^2}\right] \\
&= -\frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right] \\
A_z \sin(\phi_z) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))\,n_z(0)+\tau\gamma w_z(0)}{\omega_1^2-\omega_2^2}\right] = 0 \\
A_t \sin(\phi_t) &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))\,n_t(0)+\tau\gamma w_t(0)}{\omega_1^2-\omega_2^2}\right] \\
&= -\frac{\kappa}{\omega_1}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
B_x \sin(\theta_x) &= -\frac{\kappa}{\omega_2}\left[\frac{\left(\omega_1^2-(\kappa^2+\tau^2)\right) n_x(0)+\tau\gamma w_x(0)}{\omega_1^2-\omega_2^2}\right]=0 \\
B_y \sin(\theta_y) &= -\frac{\kappa}{\omega_2}\left[\frac{\left(\omega_1^2-(\kappa^2+\tau^2)\right) n_y(0)+\tau\gamma w_y(0)}{\omega_1^2-\omega_2^2}\right] \\
&= \frac{\kappa}{\omega_2}\left[\frac{\left(\omega_1^2-(\kappa^2+\tau^2)\right)}{\omega_1^2-\omega_2^2}\right] \\
B_z \sin(\theta_z) &= -\frac{\kappa}{\omega_2}\left[\frac{\left(\omega_1^2-(\kappa^2+\tau^2)\right) n_z(0)+\tau\gamma w_z(0)}{\omega_1^2-\omega_2^2}\right]=0 \\
B_t \sin(\theta_t) &= -\frac{\kappa}{\omega_2}\left[\frac{\left(\omega_1^2-(\kappa^2+\tau^2)\right) n_t(0)+\tau\gamma w_t(0)}{\omega_1^2-\omega_2^2}\right] \\
&= \frac{\kappa}{\omega_2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]
\end{aligned}
$$

We have

$$
\begin{aligned}
A_x &= \frac{\kappa^2 - \omega_2^2}{\omega_1^2 - \omega_2^2} \\
\phi_x &= 0 \text{ degrees} \\
A_y &= \frac{\kappa}{\omega_1}\left[\frac{(\omega_2^2 - (\kappa^2 + \tau^2))}{\omega_1^2 - \omega_2^2}\right] \\
\phi_y &= -90 \text{ degrees} \\
A_z &= \frac{\kappa\tau}{\omega_1^2 - \omega_2^2} \\
\phi_z &= 180 \text{ degrees} \\
A_t &= \frac{\kappa}{\omega_1}\left[\frac{\tau\gamma}{\omega_1^2 - \omega_2^2}\right] \\
\phi_t &= -90 \text{ degrees} \\
B_x &= \frac{(\kappa^2 - \omega_1^2)}{\omega_1^2 - \omega_2^2} \\
\theta_x &= 180 \text{ degrees} \\
B_y &= \frac{\kappa}{\omega_2}\left[\frac{(\omega_1^2 - (\kappa^2 + \tau^2))}{\omega_1^2 - \omega_2^2}\right] \\
\theta_y &= 90 \text{ degrees} \\
B_z &= \frac{\kappa\tau}{\omega_1^2 - \omega_2^2} \\
\theta_z &= 0 \text{ degrees} \\
B_t &= \frac{\kappa}{\omega_2}\left[\frac{\tau\gamma}{\omega_1^2 - \omega_2^2}\right] \\
\theta_t &= 90 \text{ degrees}
\end{aligned}
$$

Substituting into the expression for radial components, we have

$$
\begin{aligned}
r_x(s) &= \frac{A_x}{\omega_1}\sin(\omega_1 s+\phi_x)+\frac{B_x}{\omega_2}\sin(\omega_2 s+\theta_x) \\
&= \frac{1}{\omega_1}\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2}\sin(\omega_1 s)-\frac{1}{\omega_2}\frac{(\kappa^2-\omega_1^2)}{\omega_1^2-\omega_2^2}\sin(\omega_2 s) \\
r_y(s) &= \frac{A_y}{\omega_1}\sin(\omega_1 s+\phi_y)+\frac{B_y}{\omega_2}\sin(\omega_2 s+\theta_y) \\
&= -\frac{\kappa}{\omega_1^2}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right]\cos(\omega_1 s) \\
&\quad +\frac{\kappa}{\omega_2^2}\left[\frac{(\omega_1^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right]\cos(\omega_2 s) \\
r_z(s) &= \frac{A_z}{\omega_1}\sin(\omega_1 s+\phi_z)+\frac{B_z}{\omega_2}\sin(\omega_2 s+\theta_z) \\
&= -\frac{\kappa\tau}{\omega_1}\frac{1}{\omega_1^2-\omega_2^2}\sin(\omega_1 s)+\frac{\kappa\tau}{\omega_2}\frac{1}{\omega_1^2-\omega_2^2}\sin(\omega_2 s) \\
r_t(s) &= \frac{A_t}{\omega_1}\sin(\omega_1 s+\phi_t)+\frac{B_t}{\omega_2}\sin(\omega_2 s+\theta_y) \\
&= -\frac{\kappa}{\omega_1^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]\cos(\omega_1 s)+\frac{\kappa}{\omega_2^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]\cos(\omega_2 s)
\end{aligned}
$$

To reduce clutter, define

$$
\begin{aligned}
r_x(s) &= A\sin(\omega_1 s)+B\sin(\omega_2 s) \\
r_y(s) &= C\cos(\omega_1 s)+D\cos(\omega_2 s) \\
r_z(s) &= E\sin(\omega_1 s)+F\sin(\omega_2 s) \\
r_t(s) &= G\cos(\omega_1 s)+H\cos(\omega_2 s)
\end{aligned}
$$

with

$$
\begin{aligned}
A &= \frac{1}{\omega_1}\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2} \\
B &= -\frac{1}{\omega_2}\frac{(\kappa^2-\omega_1^2)}{\omega_1^2-\omega_2^2} \\
C &= -\frac{\kappa}{\omega_1^2}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right] \\
D &= \frac{\kappa}{\omega_2^2}\left[\frac{(\omega_1^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right] \\
E &= -\frac{\kappa\tau}{\omega_1}\frac{1}{\omega_1^2-\omega_2^2} \\
F &= \frac{\kappa\tau}{\omega_2}\frac{1}{\omega_1^2-\omega_2^2} \\
G &= -\frac{\kappa}{\omega_1^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right] \\
H &= \frac{\kappa}{\omega_2^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]
\end{aligned}
$$

Our square radius (written in a suggestive form) is then

$$
\begin{aligned}
R^2 = \; & (A^2+E^2)\sin^2(\omega_1 s)+(B^2+F^2)\sin^2(\omega_2 s)+ \\
& (C^2+G^2)\cos^2(\omega_1 s)+(D^2+H^2)\cos^2(\omega_2 s)+ \\
& 2(AB+EF)\sin(\omega_1 s)\sin(\omega_2 s)+ \\
& 2(CD+GH)\cos(\omega_1 s)\cos(\omega_2 s)
\end{aligned}
$$

We will now show that

$$
\begin{aligned}
AB+EF &= 0 \\
CD+GH &= 0 \\
A^2+E^2 &= C^2+G^2 \\
B^2+F^2 &= D^2+H^2
\end{aligned}
$$

We begin with $AB+EF=0$

$$
\begin{aligned}
AB &= \frac{1}{\omega_1}\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2}\left(-\frac{1}{\omega_2}\frac{(\kappa^2-\omega_1^2)}{\omega_1^2-\omega_2^2}\right)\\
EF &= \left(-\frac{\kappa\tau}{\omega_1}\frac{1}{\omega_1^2-\omega_2^2}\right)\left(\frac{\kappa\tau}{\omega_2}\frac{1}{\omega_1^2-\omega_2^2}\right)
\end{aligned}
$$

Add these together, and clear the common factors

$$
\begin{aligned}
\left(\kappa^2-\omega_1^2\right)\left(\kappa^2-\omega_2^2\right)+\kappa^2\tau^2 &\stackrel{?}{\equiv} 0\\
\kappa^4-(\omega_1^2+\omega_2^2)\kappa^2+\omega_1^2\omega_2^2+\kappa^2\tau^2 &\stackrel{?}{\equiv} 0
\end{aligned}
$$

Substituting for $\omega_1^2+\omega_2^2$ and $\omega_1^2 * \omega_2^2$, we get

$$
\begin{aligned}
\kappa^4-(\kappa^2+\tau^2+\gamma^2)\kappa^2+\kappa^2\gamma^2+\kappa^2\tau^2 &\stackrel{?}{\equiv} 0\\
\kappa^4-\kappa^4-\kappa^2\tau^2-\kappa^2\gamma^2+\kappa^2\gamma^2+\kappa^2\tau^2 &= 0
\end{aligned}
$$

Now we work on $CD+GH=0$

$$
\begin{aligned}
CD &= \left(-\frac{\kappa}{\omega_1^2}\left[\frac{(\omega_2^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right]\right)\left(\frac{\kappa}{\omega_2^2}\left[\frac{(\omega_1^2-(\kappa^2+\tau^2))}{\omega_1^2-\omega_2^2}\right]\right)\\
GH &= -\left(\frac{\kappa}{\omega_1^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]\right)\left(\frac{\kappa}{\omega_2^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]\right)
\end{aligned}
$$

Add these and eliminate the common factors

$$
\left(\omega_2^2-\left(\kappa^2+\tau^2\right)\right)\left(\omega_1^2-\left(\kappa^2+\tau^2\right)\right)+\tau^2\gamma^2 \stackrel{?}{\equiv} 0
$$

Again, by straightforward substitution

$$
\begin{aligned}
\omega_1^2\omega_2^2-(\kappa^2+\tau^2)(\omega_1^2+\omega_2^2)+(\kappa^2+\tau^2)^2+\tau^2\gamma^2 &\stackrel{?}{\equiv} 0\\
\kappa^2\gamma^2-(\kappa^2+\tau^2)(\kappa^2+\tau^2+\gamma^2)+(\kappa^2+\tau^2)^2+\tau^2\gamma^2 &\stackrel{?}{\equiv} 0\\
\kappa^2\gamma^2-(\kappa^2+\tau^2)^2-(\kappa^2+\tau^2)\gamma^2+(\kappa^2+\tau^2)^2+\tau^2\gamma^2 &= 0
\end{aligned}
$$

We now begin to show $A^2+E^2=C^2+G^2$, and evaluate the A^2+E^2 term.

$$
\begin{aligned}
A^2+E^2 &= \left(\frac{1}{\omega_1}\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2}\right)^2+\left(-\frac{\kappa\tau}{\omega_1}\frac{1}{\omega_1^2-\omega_2^2}\right)^2 \\
&= \frac{\omega_1^2\left(\kappa^2-\omega_2^2\right)^2+\omega_1^2\kappa^2\tau^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2} \\
C^2+G^2 &= \left(-\frac{\kappa}{\omega_1^2}\frac{\left(\omega_2^2-\left(\kappa^2+\tau^2\right)\right)}{\omega_1^2-\omega_2^2}\right)^2+\left(-\frac{\kappa}{\omega_1^2}\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right)^2 \\
&= \frac{\kappa^2\left(\omega_2^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2}
\end{aligned}
$$

Eliminate the common denominator, and substitute for ω_1 and ω_2, collect terms, and we find left and right sides equal.

$$
\begin{aligned}
\frac{\omega_1^2\left(\kappa^2-\omega_2^2\right)^2+\omega_1^2\kappa^2\tau^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2} &\overset{?}{=} \frac{\kappa^2\left(\omega_2^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2} \\
\omega_1^2\left(\kappa^2-\omega_2^2\right)^2+\omega_1^2\kappa^2\tau^2 &\overset{?}{=} \kappa^2\left(\omega_2^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2
\end{aligned}
$$

By simple substitution for ω_1^2 and ω_2^2, both sides are found equal to

$$
\begin{aligned}
&\frac{\kappa^2}{2}\left(\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2\right) \\
&+\left(\kappa^2+\tau^2-\gamma^2\right)\sqrt{\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2}
\end{aligned}
$$

In a similar fashion, we form the expressions

$$
\begin{aligned}
B^2+F^2 &= \left(-\frac{1}{\omega_2}\frac{\left(\kappa^2-\omega_1^2\right)}{\omega_1^2-\omega_2^2}\right)^2+\left(\frac{\kappa\tau}{\omega_2}\frac{1}{\omega_1^2-\omega_2^2}\right)^2 \\
&= \frac{\omega_2^2\left(\kappa^2-\omega_1^2\right)^2+\omega_2^2\kappa^2\tau^2}{\omega_2^4\left(\omega_1^2-\omega_2^2\right)^2} \\
D^2+H^2 &= \left(\frac{\kappa}{\omega_2^2}\left[\frac{\left(\omega_1^2-\left(\kappa^2+\tau^2\right)\right)}{\omega_1^2-\omega_2^2}\right]\right)^2+\left(\frac{\kappa}{\omega_2^2}\left[\frac{\tau\gamma}{\omega_1^2-\omega_2^2}\right]\right)^2 \\
&= \frac{\kappa^2\left(\omega_1^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2}{\omega_2^4\left(\omega_1^2-\omega_2^2\right)^2}
\end{aligned}
$$

Eliminate the common denominator, and substitute for ω_1 and ω_2, collect terms, and we find left and right sides equal.

$$\begin{aligned}
\frac{\omega_2^2\left(\kappa^2-\omega_1^2\right)^2+\omega_2^2\kappa^2\tau^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2} &\stackrel{?}{=} \frac{\kappa^2\left(\omega_1^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2}{\omega_1^4\left(\omega_1^2-\omega_2^2\right)^2} \\
\omega_2^2\left(\kappa^2-\omega_1^2\right)^2+\omega_2^2\kappa^2\tau^2 &\stackrel{?}{=} \kappa^2\left(\omega_1^2-\left(\kappa^2+\tau^2\right)\right)^2+\kappa^2\tau^2\gamma^2
\end{aligned}$$

Both of these sides are equal to

$$\begin{aligned}
&\frac{\kappa^2}{2}\left(\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2\right) \\
&-\left(\kappa^2+\tau^2-\gamma^2\right)\sqrt{\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2}
\end{aligned}$$

We are now ready for the closed form solution for the radius squared. Using the simplest terms,

$$\begin{aligned}
R^2 &= A^2+E^2+B^2+F^2 \\
A^2 &= \frac{\left(\kappa^2-\omega_2^2\right)^2}{\omega_2^2\left(\omega_1^2-\omega_2^2\right)^2} \\
E^2 &= \frac{\kappa^2\tau^2}{\omega_2^2\left(\omega_1^2-\omega_2^2\right)^2} \\
B^2 &= \frac{\left(\kappa^2-\omega_1^2\right)^2}{\omega_1^2\left(\omega_1^2-\omega_2^2\right)^2} \\
F^2 &= \frac{\kappa^2\tau^2}{\omega_1^2\left(\omega_1^2-\omega_2^2\right)^2}
\end{aligned}$$

Doing the paperwork,

$$
\begin{aligned}
R^2 &= \frac{\left(\kappa^2-\omega_2^2\right)^2}{\omega_2^2\left(\omega_1^2-\omega_2^2\right)^2}+\frac{\kappa^2\tau^2}{\omega_2^2\left(\omega_1^2-\omega_2^2\right)^2}+\frac{\left(\kappa^2-\omega_1^2\right)^2}{\omega_1^2\left(\omega_1^2-\omega_2^2\right)^2}+\frac{\kappa^2\tau^2}{\omega_1^2\left(\omega_1^2-\omega_2^2\right)^2} \\
&= \frac{\omega_2^2\left(\kappa^2-\omega_2^2\right)^2+\omega_1^2\left(\kappa^2-\omega_1^2\right)^2+\kappa^2\tau^2\left(\omega_1^2+\omega_2^2\right)}{\omega_1^2\omega_2^2\left(\omega_1^2-\omega_2^2\right)^2} \\
&= \frac{\omega_2^2\left(\kappa^2-\omega_2^2\right)^2+\omega_1^2\left(\kappa^2-\omega_1^2\right)^2+\kappa^2\tau^2\left(\omega_1^2+\omega_2^2\right)}{\kappa^2\gamma^2\left(\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2\right)} \\
&= \frac{\left(\tau^2+\gamma^2\right)\left(\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2\right)}{\kappa^2\gamma^2\left(\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2\right)} \\
&= \frac{\tau^2+\gamma^2}{\kappa^2\gamma^2} \\
&= \frac{1}{\kappa^2}+\frac{1}{\kappa^2}\left(\frac{\tau^2}{\gamma^2}\right) \\
R^2 &= \frac{1}{\kappa^2}\left(1+\frac{\tau^2}{\gamma^2}\right)=\frac{\tau^2+\gamma^2}{\kappa^2\gamma^2}
\end{aligned}
$$

Single Frequencies in Orthogonal Planes

Looking at the standard form for the coordinates or tangents, we can see that we can rotate our coordinate system to achieve single frequency components in orthogonal planes. I am continuing to use the position coordinates in this demonstation. However, doing the same to the unit tangent component achieves the same result, and is probably wiser for numerical results.

Start with the coordinate representation

$$
\begin{aligned}
x &= A\sin(\omega_1 s)+B\sin(\omega_2 s) \\
y &= C\cos(\omega_1 s)+D\cos(\omega_2 s) \\
z &= E\sin(\omega_1 s)+F\sin(\omega_2 s) \\
t &= G\cos(\omega_1 s)+H\cos(\omega_2 s)
\end{aligned}
$$

with the relationships shown above of

$$\begin{aligned} AB + EF &= 0 \\ CD + GH &= 0 \\ A^2 + E^2 &= C^2 + G^2 \\ B^2 + F^2 &= D^2 + H^2 \end{aligned}$$

We can form linear combinations to eliminate frequency components.

$$\begin{aligned} Ax + Ez &= (A^2 + E^2)\sin(\omega_1 s) \\ Bx + Fz &= (B^2 + F^2)\sin(\omega_2 s) \\ Cy + Gt &= (C^2 + G^2)\cos(\omega_1 s) \\ Dy + Ht &= (D^2 + H^2)\cos(\omega_2 s) \end{aligned}$$

We can normalize these to rotations easily showing radial components

$$\begin{aligned} X &= \frac{Ax + Ez}{\sqrt{A^2 + E^2}} = \sqrt{A^2 + E^2}\sin(\omega_1 s) \\ Y &= \frac{Cy + Gt}{\sqrt{C^2 + G^2}} = \sqrt{C^2 + G^2}\cos(\omega_1 s) \\ Z &= \frac{Bx + Fz}{\sqrt{B^2 + F^2}} = \sqrt{B^2 + F^2}\sin(\omega_2 s) \\ T &= \frac{Dy + Ht}{\sqrt{D^2 + H^2}} = \sqrt{D^2 + H^2}\cos(\omega_2 s) \end{aligned}$$

We see we have circulation in the XY plane at frequency ω_1 and radius $r_1 = \sqrt{A^2 + E^2}$. We have circulation in the ZT plane at frequency ω_2 and radius $r_2 = \sqrt{C^2 + G^2}$.

Substituting values, we find

$$\begin{aligned} r_1^2 &= A^2 + E^2 \\ &= \frac{\left(\kappa^2 - \omega_2^2\right)^2 + \kappa^2\tau^2}{\omega_1^2\left(\omega_1^2 - \omega_2^2\right)^2} = \frac{\kappa^2 - \omega_2^2}{\omega_1^2\left(\omega_1^2 - \omega_2^2\right)} \\ r_2^2 &= B^2 + F^2 \\ &= \frac{\left(\kappa^2 - \omega_1^2\right)^2 + \kappa^2\tau^2}{\omega_2^2\left(\omega_1^2 - \omega_2^2\right)^2} = \frac{-\kappa^2 + \omega_1^2}{\omega_2^2\left(\omega_1^2 - \omega_2^2\right)} \end{aligned}$$

Formulas for Curvatures given Radii and Frequencies

The above work has progressed from curvature, torsion and lift, and found frequencies and radii. The system has three degrees of freedom, and we could just as well specify any three independent parameters, and then find our curvatures. For example, we could specify major and minor radii and a frequency, or ratios of radii and two frequencies, and so on. To develop these formulas, I begin by specifying the curves using radii and frequencies.

$$\begin{aligned} x &= r_1 \cos(\omega_1 s) \\ y &= r_1 \sin(\omega_1 s) \\ z &= r_2 \cos(\omega_2 s) \\ t &= r_2 \sin(\omega_2 s) \end{aligned}$$

Take our derivatives with respect to s.

$$\begin{aligned} u_x &= -r_1\omega_1 \sin(\omega_1 s) = -\omega_1 y \\ u_y &= r_1\omega_1 \cos(\omega_1 s) = \omega_1 x \\ u_z &= -r_2\omega_2 \sin(\omega_2 s) = -\omega_2 t \\ u_t &= r_2\omega_2 \cos(\omega_2 s) = \omega_2 z \end{aligned}$$

From the unit magnitude of the tangent, we have

$$r_1^2\omega_1^2 + r_2^2\omega_2^2 = 1$$

We can use this equation to determine one item given the other three. For example, given two radii and a ratio of frequencies, we find

$$r_1^2 + r_2^2\frac{\omega_2^2}{\omega_1^2} = \frac{1}{\omega_1^2}$$

specifies frequencies, and so forth. From the unit magnitude above, can define a characteristic angle θ, with $\cos\theta = r_1\omega_1$ and $\sin\theta = r_2\theta_2$. Taking derivatives again to find the curvature relationship

$$\frac{du_x}{ds} = \kappa n_x = -r_1\omega_1^2\cos(\omega_1 s)$$
$$\frac{du_y}{ds} = \kappa n_y = -r_1\omega_1^2\sin(\omega_1 s)$$
$$\frac{du_z}{ds} = \kappa n_z = -r_2\omega_2^2\cos(\omega_2 s)$$
$$\frac{du_t}{ds} = \kappa n_t = -r_2\omega_2^2\sin(\omega_2 s)$$

From this, we find the magnitude of the curvature

$$\begin{aligned}\kappa^2 &= r_1^2\omega_1^4 + r_2^2\omega_2^4 \\ &= \omega_1^2\cos^2\theta + \omega_2^2\sin^2\theta\end{aligned}$$

Seeing another opportunity for angle specification, we can write

$$\begin{aligned}\frac{\kappa^2}{\omega_1^2+\omega_2^2} &= \frac{\omega_1^2}{\omega_1^2+\omega_2^2}\cos^2\theta + \frac{\omega_2^2}{\omega_1^2+\omega_2^2}\sin^2\theta \\ &= \cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\end{aligned}$$

Taking the derivative of the normal, and solving for torsion, we get

$$\begin{aligned}\tau^2 &= \frac{1}{\kappa^2}\left[\left(\kappa^2-\omega_1^2\right)^2(r_1\omega_1)^2 + \left(\kappa^2-\omega_2^2\right)^2(r_2\omega_2)^2\right] \\ \tau^2\kappa^2 &= \left(\kappa^2-\omega_1^2\right)^2(r_1\omega_1)^2 + \left(\kappa^2-\omega_2^2\right)^2(r_2\omega_2)^2 \\ &= \left(\kappa^2-\omega_1^2\right)^2\cos^2\theta + \left(\kappa^2-\omega_2^2\right)^2\sin^2\theta\end{aligned}$$

Further solving for the binormal and trinormal, we find the lift is

$$\begin{aligned}\gamma^2 &= \frac{\left(\kappa^2+\tau^2-\omega_1^2\right)^2}{\kappa^2\tau^2}\omega_1^4 r_1^2 + \frac{\left(\kappa^2+\tau^2-\omega_2^2\right)^2}{\kappa^2\tau^2}\omega_2^4 r_2^2 \\ \gamma^2\tau^2\kappa^2 &= \left(\kappa^2+\tau^2-\omega_1^2\right)^2\omega_1^2\cos^2\theta + \left(\kappa^2+\tau^2-\omega_2^2\right)^2\omega_2^2\sin^2\theta\end{aligned}$$

Angles Associated with Curvatures

Each of the curvatures above is composed from two orthogonal terms. We can associate an angle with each of these curvatures.

$$
\begin{aligned}
1 &= r_1^2\omega_1^2 + r_2^2\omega_2^2 \\
\theta_u &= \tan^{-1}\left(\frac{r_2\omega_2}{r_1\omega_1}\right) \\
\kappa^2 &= r_1^2\omega_1^4 + r_2^2\omega_2^4 \\
\theta_\kappa &= \tan^{-1}\left(\frac{\omega_2}{\omega_1}\tan\theta_u\right) \\
\tau^2 &= \frac{1}{\kappa^2}\left[\left(\kappa^2-\omega_1^2\right)^2(r_1\omega_1)^2 + \left(\kappa^2-\omega_2^2\right)^2(r_2\omega_2)^2\right] \\
\theta_\tau &= \tan^{-1}\left(\frac{\kappa^2-\omega_2^2}{\kappa^2-\omega_1^2}\tan\theta_u\right) \\
\gamma^2 &= \frac{\left(\kappa^2+\tau^2-\omega_1^2\right)^2}{\kappa^2\tau^2}r_1^2\omega_1^4 + \frac{\left(\kappa^2+\tau^2-\omega_2^2\right)^2}{\kappa^2\tau^2}r_2^2\omega_2^4 \\
\theta_\gamma &= \tan^{-1}\left(\frac{\kappa^2+\tau^2-\omega_2^2}{\kappa^2+\tau^2-\omega_1^2}\tan\theta_\kappa\right)
\end{aligned}
$$

Summary of General Results

Curves of constant curvature, torsion and lift trace curves on the surface of a hypersphere with fixed radius.

$$
R^2 = \frac{\tau^2+\gamma^2}{\kappa^2\gamma^2}
$$

The trajectory parameterized by pathlength (self-history) has two frequency components and two orthogonal radii.

$$
\begin{aligned}
\omega_1 &= \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)+\sqrt{\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2}}{2}} \\
\omega_2 &= \sqrt{\frac{(\kappa^2+\tau^2+\gamma^2)-\sqrt{\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2}}{2}}
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= \frac{\sqrt{\left(\kappa^2-\omega_2^2\right)^2+\kappa^2\tau^2}}{\omega_1\left(\omega_1^2-\omega_2^2\right)} = \frac{1}{\omega_1}\sqrt{\frac{\kappa^2-\omega_2^2}{\omega_1^2-\omega_2^2}} \\
r_2 &= \frac{\sqrt{\left(\kappa^2-\omega_1^2\right)^2+\kappa^2\tau^2}}{\omega_2\left(\omega_1^2-\omega_2^2\right)} = \frac{1}{\omega_2}\sqrt{\frac{\omega_1^2-\kappa^2}{\omega_1^2-\omega_2^2}}
\end{aligned}
$$

We have the relationship

$$
\left(r_1\omega_1\right)^2+\left(r_2\omega_2\right)^2 = 1
$$

The product of these two frequencies the also the product of curvature and lift

$$
\omega_1 * \omega_2 = \kappa\gamma
$$

The ratio of these two frequencies is usually not a rational number.

$$
\frac{\omega_1}{\omega_2} = \frac{\left(\kappa^2+\tau^2+\gamma^2\right)+\sqrt{\left(\kappa^2+\tau^2+\gamma^2\right)^2-4\kappa^2\gamma^2}}{2\kappa\gamma}
$$

Consequently, the curve usually covers a band on the foursphere. However, for specific values of curvature, torsion and lift, we can have the two frequencies harmonically locked, resulting in a filament, rather than a surface on the the foursphere.

Since the curves of constant curvature and lift orbit at two radii and frequencies in orthogonal planes, we can write formulas relating curvatures, radii and frequency.

$$
\begin{aligned}
1 &= r_1^2\omega_1^2+r_2^2\omega_2^2 \\
\kappa^2 &= r_1^2\omega_1^4+r_2^2\omega_2^4 \\
\tau^2 &= \frac{1}{\kappa^2}\left[\left(\kappa^2-\omega_1^2\right)^2\left(r_1\omega_1\right)^2+\left(\kappa^2-\omega_2^2\right)^2\left(r_2\omega_2\right)^2\right] \\
\gamma^2 &= \frac{\left(\kappa^2+\tau^2-\omega_1^2\right)^2}{\kappa^2\tau^2}r_1^2\omega_1^4+\frac{\left(\kappa^2+\tau^2-\omega_2^2\right)^2}{\kappa^2\tau^2}r_2^2\omega_2^4
\end{aligned}
$$

Given three curvatures, we can calculate frequencies, then radii, and we can then align our coordinates system for easy calculation of our basis vectors.

$$\begin{aligned}
x &= r_1 \cos(\omega_1 s) \\
y &= r_1 \sin(\omega_1 s) \\
z &= r_2 \cos(\omega_2 s) \\
t &= r_2 \sin(\omega_2 s)
\end{aligned}$$

$$\begin{aligned}
\tilde{u} &= \frac{d\tilde{r}}{ds} \\
u_x &= -r_1\omega_1 \sin(\omega_1 s) \\
u_y &= r_1\omega_1 \cos(\omega_1 s) \\
u_z &= -r_2\omega_2 \sin(\omega_2 s) \\
u_t &= r_2\omega_2 \cos(\omega_2 s)
\end{aligned}$$

$$\begin{aligned}
\tilde{n} &= \frac{1}{\kappa}\frac{d\tilde{u}}{ds} \\
n_x &= \left(-r_1\omega_1^2/\kappa\right) \cos(\omega_1 s) \\
n_y &= \left(-r_1\omega_1^2/\kappa\right) \sin(\omega_1 s) \\
n_z &= \left(-r_2\omega_2^2/\kappa\right) \cos(\omega_2 s) \\
n_t &= \left(-r_2\omega_2^2/\kappa\right) \sin(\omega_2 s)
\end{aligned}$$

$$\begin{aligned}
\tilde{b} &= \frac{1}{\tau}\left(\frac{d\tilde{n}}{ds} + \kappa\tilde{u}\right) \\
b_x &= \left[\left(r_1\omega_1^3/\kappa - \kappa\omega_1 r_1\right)/\tau\right] \sin(\omega_1 s) \\
b_y &= \left[\left(-r_1\omega_1^3/\kappa + \kappa\omega_1 r_1\right)/\tau\right] \cos(\omega_1 s) \\
b_z &= \left[\left(r_2\omega_2^3/\kappa - \kappa\omega_2 r_2\right)/\tau\right] \sin(\omega_2 s) \\
b_t &= \left[\left(-r_2\omega_2^3/\kappa + \kappa\omega_2 r_2\right)/\tau\right] \cos(\omega_2 s)
\end{aligned}$$

$$
\begin{aligned}
\tilde{w} &= \frac{1}{\gamma}\left(\frac{d\tilde{b}}{ds} + \tau\tilde{n}\right) \\
w_x &= \left[\left(\left[\left(r_1\omega_1^4/\kappa - \kappa\omega_1^2 r_1\right)/\tau\right] - \tau\omega_1^2 r_1/\kappa\right)/\gamma\right]\cos(\omega_1 s) \\
w_y &= \left[\left(\left[\left(r_1\omega_1^4/\kappa - \kappa\omega_1^2 r_1\right)/\tau\right] - \tau\omega_1^2 r_1/\kappa\right)/\gamma\right]\sin(\omega_1 s) \\
w_z &= \left[\left(\left[\left(r_2\omega_2^4/\kappa - \kappa\omega_2^2 r_2\right)/\tau\right] - \tau\omega_2^2 r_2/\kappa\right)/\gamma\right]\cos(\omega_2 s) \\
w_t &= \left[\left(\left[\left(r_2\omega_2^4/\kappa - \kappa\omega_2^2 r_2\right)/\tau\right] - \tau\omega_2^2 r_2/\kappa\right)/\gamma\right]\sin(\omega_2 s)
\end{aligned}
$$