The Antisymmetric Symbol - The Great Normalizer

Kurt Nalty

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The Antisymmetric Symbol

The generalized, antisymmetric symbol ϵ is usually introduced and defined in three dimensions associated with the vector cross product, as a function of three indices, which range in value from 1 to 3. Any even permutation of 1,2,3 yields +1, any odd permutation is -1, and any combination which repeats an index is 0

$$\begin{aligned}
\epsilon(1,2,3) &= 1 \\
\epsilon(2,3,1) &= 1 \\
\epsilon(3,1,2) &= 1 \\
\epsilon(3,2,1) &= -1 \\
\epsilon(2,1,3) &= -1 \\
\epsilon(1,3,2) &= -1
\end{aligned}$$

Writing vectors in tensor notation, the cross product is then defined as $C_i = A_j B_k \epsilon_i^{jk}$.

As we extend to higher dimensions, the symbol takes more indices, and maintains the rule that any even permutation of unique, ordered indices is 1, any odd permutation is -1, and any duplicated indices go to zero.

$$\epsilon_{ijklm} = -\epsilon_{jiklm} = 1$$

The contraction (dot product) of any symmetric expression with the antisymmetric symbol is always zero. $A_j A_k \epsilon_i^{jk} = 0$.

In my younger days, this was as far as I thought. I simply accepted the symbol as a standard tool without investigation. In these following paragraphs, I examine and admire this very useful symbol in a bit more detail.

Calculating the Antisymmetric Symbol

The first step in my work was to build a formula for this symbol, rather than using a lookup table approach.

Two Dimensions

Starting with two indices, the antisymmetric symbol is simple the difference between these indices. The inherent antisymmetry of subtraction gives us our polarization properties. For the indices i and j having values of 1 and 2, we have

$$\begin{aligned} \epsilon_i^j &= j - i \\ \epsilon_1^1 &= 0 \\ \epsilon_1^2 &= 1 \\ \epsilon_2^1 &= -1 \\ \epsilon_2^2 &= 0 \end{aligned}$$

This symbol creates a normal from a planar vector, with the same magnitude as the original vector.

$$\begin{aligned} B_i &= A_j \epsilon_i^j \\ B_1 &= A_1 \epsilon_1^1 + A_2 \epsilon_1^2 = A_2 \\ B_2 &= A_1 \epsilon_2^1 + A_2 \epsilon_2^2 = -A_1 \end{aligned}$$

Three Dimensions

Taking the product of all possible differences gives us the generalized antisymmetric polarity behavior. We need a normalizing factor to get unit magnitude for the non-zero members.

$$\begin{aligned} \epsilon_i^{jk} &= \frac{(k-j)*(k-i)*(j-i)}{2} \\ &= \frac{(k-j)*(k-i)}{2!}\epsilon_i^j \end{aligned}$$

Taking the product of two vectors, and the three dimensional anti-symmetric tensor, we get the cross product, with magnitude equal the area of the parallelogram formed by the two vectors, and a direction normal to the two vectors.

Four Dimensions

Continuing the suggestive process above, we write

$$\begin{aligned} \epsilon_i^{jkl} &= \frac{(l-k)*(l-j)*(l-i)*(k-j)*(k-i)*(j-i)}{3!2!1!} \\ &= \frac{(l-k)*(l-j)*(l-i)}{3!}\epsilon_i^{jk} \end{aligned}$$

The product of three vectors with the above gives a rhomboid volume made by the three vectors, and a normal direction to all three.

Comments

The above formula generalizes to any integer dimensional space. Further, since the terms are differences, it does not matter if we use indices $1, 2, 3...$ or $0, 1, 2....$ Furthermore, since the normalization factors all divide out anyway, a software routine can just use the ordered differences and toggle a sign bit.

The antisymmetric symbol can be used to implement wedge products, calculate hypervolumes, and calculate determinants.