Transmission Lines, Stacked Insulated Washers Lines, Tesla Coils and the Telegrapher's Equation

Kurt Nalty

August 1, 2008

Abstract

Tesla coils have more in common with transmission lines than transformers. This note presents the common model (telegrapher's equation) for the parallel conductor transmission line, derives the similar telegrapher's equations for the rotated transmission line comprised of insulated stacked washers, then presents and solves the single layer solenoid model for the Tesla coil.

1 Standard Transmission Lines

The standard transmission line consists of two parallel conductors separated by a dielectric. These conductors have a distributed inductance per length (L'), resistance per length (R'), and are shunted by a distribute capacitance per length (C') and leakage conductance (G').

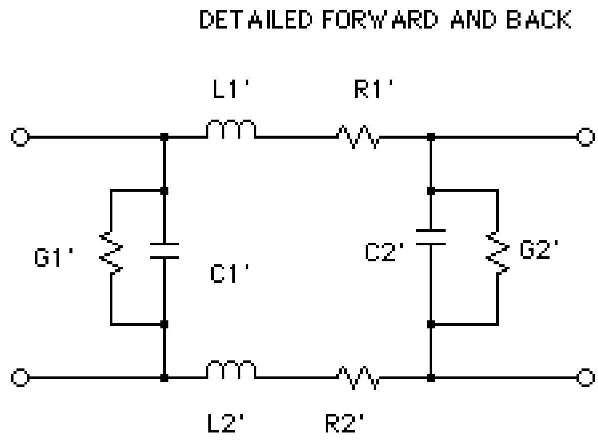

Figure 1 - Detailed Conventional Transmission Line

In general, L1' and L2' are not equal, likewise the other terms. However, from the point of view of lazy analysis, we can treat the cable with these items lumped together (series L1' and L2', series R1' and R2', parallel C1' and C2', parallel G1' and G2'), with a simplified model shown in figure 2.

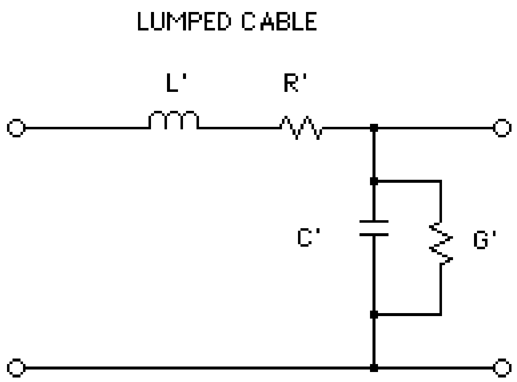

Figure 2 - Lumped Conventional Transmission Line

The transmission line models as a low pass filter. The voltage and current on this line vary as functions of distance and time.

The voltage gradient along the cable is given by

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial L}{\partial x}\frac{\partial I}{\partial t} + \frac{\partial R}{\partial x} I \\ &= L'\frac{\partial I}{\partial t} + R' I\end{aligned}$$

Likewise, the shunting elements lead to current loss

$$\begin{aligned}\frac{\partial I}{\partial x} &= \frac{\partial G}{\partial x} V + \frac{\partial C}{\partial x}\frac{\partial V}{\partial t} \\ &= C'\frac{\partial V}{\partial t} + G' V\end{aligned}$$

These two coupled partial differential equations are the Telegrapher's Equations.

$$\begin{aligned}\frac{\partial V}{\partial x} &= L'\frac{\partial I}{\partial t} + R' I \\ \frac{\partial I}{\partial x} &= C'\frac{\partial V}{\partial t} + G' V\end{aligned}$$

We can eliminate the coupling by differentiating the top equation by x, the bottom by t, and using substitution.

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} &= L'\frac{\partial^2 I}{\partial x \partial t} + R'\frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial x} &= C'\frac{\partial V}{\partial t} + G' V \\ \frac{\partial^2 I}{\partial x \partial t} &= C'\frac{\partial^2 V}{\partial t^2} + G'\frac{\partial V}{\partial t} \\ \frac{\partial^2 V}{\partial x^2} &= L'\left(C'\frac{\partial^2 V}{\partial t^2} + G'\frac{\partial V}{\partial t}\right) + R'\left(C'\frac{\partial V}{\partial t} + G' V\right)\end{aligned}$$

or

$$\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2} + (L'G' + C'R')\frac{\partial V}{\partial t} + R'G'V \tag{1}$$

In a similar fashion, the current equation is given by

$$\frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2} + (L'G' + C'R')\frac{\partial I}{\partial t} + R'G'I \tag{2}$$

These equations reduce to the wave equation when R' and G' are both low. The speed of propagation is

$$v_z = 1.0/\sqrt{L'C'}$$

The energy storage splits equally between capacitive and inductive storage.

$$E' = \frac{1}{2}L'I^2 = \frac{1}{2}C'V^2$$

From this, we find the ratio of voltage and current, which defines the characteristic line impedance.

$$\begin{aligned}
\frac{1}{2}L'I^2 &= \frac{1}{2}C'V^2 \\
\frac{C'V^2}{L'I^2} &= 1 \\
\frac{V^2}{I^2} &= \frac{L'}{C'} \\
Z &= \frac{V}{I} = \sqrt{\frac{L'}{C'}}
\end{aligned}$$

1.1 L' and C' examples

For coax cable,

$$\begin{aligned}
L' &= \frac{\mu}{2\pi}\ln\frac{R}{r} \\
C' &= \frac{2\pi\epsilon}{\ln(R/r)} \\
Z &= \sqrt{\frac{\mu}{\epsilon}}\frac{\ln(R/r)}{2\pi} \\
v &= \frac{c}{\sqrt{\mu_r\epsilon_r}}
\end{aligned}$$

For twinlead cable,

$$L' = \frac{\mu}{\pi}\ln\left(\frac{d}{r}\right)$$

$$
\begin{aligned}
C' &= \frac{\pi\epsilon}{\ln(d/r)} \\
Z &= \sqrt{\frac{\mu}{\epsilon}} \left(\frac{\ln d/r}{\pi} \right) \\
v &= \frac{c}{\sqrt{\mu_r \epsilon_r}}
\end{aligned}
$$

1.2 Separation of Variables Solution

The telegrapher's differential equation is commonly solved by the separation of variables technique. Substitute $V = X(x) * T(t)$

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &= L'C' \frac{\partial^2 V}{\partial t^2} + (L'G' + C'R') \frac{\partial V}{\partial t} + R'G'V \\
\frac{\partial^2 X(x)}{\partial x^2} T(t) &= X(x) \left(L'C' \frac{\partial^2 T(t)}{\partial t^2} + (L'G' + C'R') \frac{\partial T(t)}{\partial t} + R'G'T(t) \right) \\
\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} &= \frac{1}{T(t)} \left(L'C' \frac{\partial^2 T(t)}{\partial t^2} + (L'G' + C'R') \frac{\partial T(t)}{\partial t} + R'G'T(t) \right)
\end{aligned}
$$

This last equation has the two sides equal to an arbitrary constant.

$$
\begin{aligned}
K &= \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} \\
&= \frac{1}{T(t)} \left(L'C' \frac{\partial^2 T(t)}{\partial t^2} + (L'G' + C'R') \frac{\partial T(t)}{\partial t} + R'G'T(t) \right)
\end{aligned}
$$

For K positive, $X(x) = C_1 \exp(x\sqrt{K}) + C_2 \exp(-x\sqrt{K})$. For K negative, the spatial solution is $X(x) = C_1 \sin x\sqrt{-K} + C_2 \cos x\sqrt{-K}$, where the wavenumber is $k = \sqrt{-K}$.

The temporal equation has similar solutions when R' and G' are small, with the exception that the frequencies are scaled by $\sqrt{L'C'}$. When R' and G' are not negligible, the solutions are damped (due to $L'G' + C'R' > 0$) sinusoids, whose complex frequencies are given by the generally complex roots of the equation

$$
L'C's^2 + (L'G' + C'R')\, s + (R'G' - K) = 0
$$

We then have

$$
\begin{aligned}
T(t) &= C_3 \exp s_1 t + C_4 \exp s_2 t \\
X(x) &= C_1 \sin \omega x + C_2 \cos \omega x
\end{aligned}
$$

The general solution is then the sum over all frequencies ω to match boundary conditions.

1.2.1 Group and Phase Velocity For Transmission Line

For the low loss case, we have the wavenumber $k = \sqrt{-K}$ and the angular frequency $\omega = \sqrt{-K/(L'C')} = k/\sqrt{L'C'}$.

We have phase velocity and group velocity equal

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{L'C'}}$$
$$v_g = \frac{\partial\omega}{\partial k} = \frac{1}{\sqrt{L'C'}}$$

1.3 Conventional Transmission Line Summary

- Low Pass Filter Dominant Characteristic (Higher Passbands also exist)
- Velocity of Propagation $v = 1.0/\sqrt{L'C'}$ is subluminal
- Characteristic Impedance $Z = \sqrt{L'/C'}$

2 Stacked Insulated Washer Transmission Lines

The standard transmission line consists of two parallel conductors separated by a dielectric. The stacked insulated washer line rotates this four port network 90 degrees, to be primarily capacitive coupled forward path shunted by inductance.

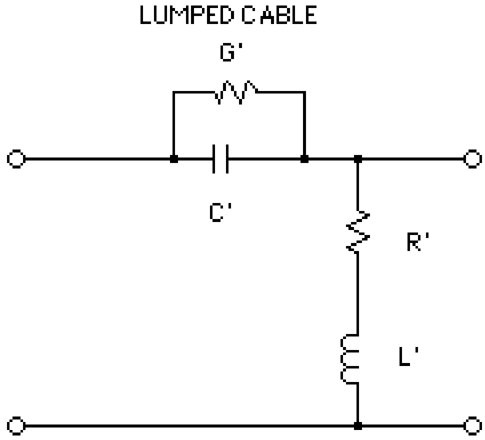

Figure 2 - Stacked Washer Transmission Line

Due to the change of orientation of the elements, we will have a change in the parameters of interest. While the illustation above is labeled with C', for example, the real parameter of interest will be the gradient of elastance, $\frac{\partial}{\partial x}\frac{1}{C}$. This is nothing more than conductance versus resistance type of issues. The reciprocal of capacitance is elastance, the reciprocal of resistance is conductance, and the reciprocal of inductance, I choose to call defluxtance, as reluctance has a specific (and slightly different) meaning for magnetic circuits.

The Stacked Washer transmission line is inherently a high pass filter. The time constants of the capacitive and inductive sections are

$$\tau_c = \frac{\frac{\partial}{\partial x}\left(\frac{1}{G}\right)}{\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}$$
$$\tau_L = \frac{\frac{\partial}{\partial x}\left(\frac{1}{R}\right)}{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)}$$

2.1 Examples of $\frac{\partial}{\partial x}\frac{1}{C}$ and $\frac{\partial}{\partial x}\frac{1}{L}$

The gradients of elastance and defluxtance are found in a fashion similar to L' and C' calculations from above.

For the case of parallel plate capacitors,

$$\begin{aligned} C &= \frac{\epsilon A}{d} \\ \frac{1}{C} &= \frac{d}{\epsilon A} \\ \frac{\partial}{\partial x}\left(\frac{1}{C}\right) &= \frac{1}{\epsilon A} \end{aligned}$$

For the case of twinlead,

$$\begin{aligned} C &= \frac{\pi \epsilon l}{\ln\left(\frac{d}{r}\right)} \\ \frac{1}{C} &= \frac{\ln\left(\frac{d}{r}\right)}{\pi \epsilon l} \\ \frac{\partial}{\partial x}\left(\frac{1}{C}\right) &= \frac{1}{\pi \epsilon l d} \end{aligned}$$

$$\begin{aligned} L &= \frac{\mu l}{\pi} \ln\left(\frac{d}{r}\right) \\ \frac{1}{L} &= \frac{\pi}{\mu l} \frac{1}{\ln\left(\frac{d}{r}\right)} \\ \frac{\partial}{\partial x}\left(\frac{1}{L}\right) &= -\frac{\pi}{\mu l d}\left(\frac{1}{\ln\left(\frac{d}{r}\right)}\right)^2 \end{aligned}$$

As the twinlead separation increases, the inductance per foot increases due to lower mutual inductance compensation. This, in turn, leads to decreasing defluxtance with parallel separation.

2.2 Differential Equations for the Stacked Washer Transmission Line

In a fashion akin to the Telegrapher's Equation, we derive the equations for the stacked washer transmission line.

The voltage across the capacitor is $V = (1/C)Q$. Taking the gradient in x, we find

$$\begin{aligned} \frac{\partial V}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{C}\right) Q \\ \frac{\partial^2 V}{\partial x \partial t} &= \frac{\partial}{\partial x}\left(\frac{1}{C}\right) I_c \\ I_c &= \frac{\frac{\partial^2 V}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{C}\right)} \end{aligned}$$

The current through the shunting conductivity is

$$I_G = \frac{\frac{\partial V}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{G}\right)}$$

Our total current is $I = I_G + I_c$.

$$I = \frac{\frac{\partial V}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{G}\right)} + \frac{\frac{\partial^2 V}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}$$

The shunted current through the inductor is $I_L = \Phi/L$. Taking our gradient in x, we get

$$\begin{aligned}
I_L &= \Phi/L \\
\frac{\partial I}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{L}\right)\Phi \\
\frac{\partial^2 I}{\partial x \partial t} &= \frac{\partial}{\partial x}\left(\frac{1}{L}\right)V_L \\
V_L &= \frac{\frac{\partial^2 I}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)}
\end{aligned}$$

Finding the voltage across the series resistor, we have

$$\begin{aligned}
I_R &= \frac{1}{R}V_R \\
\frac{\partial}{\partial x}I_R &= \frac{\partial}{\partial x}I = \frac{\partial}{\partial x}\left(\frac{1}{R}\right)V_R \\
V_R &= \frac{\frac{\partial I}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{R}\right)}
\end{aligned}$$

Our voltage across this branch is $V = V_L + V_R$.

$$V = \frac{\frac{\partial I}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{R}\right)} + \frac{\frac{\partial^2 I}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)}$$

We now have our set of Telegrapher's Equations for the stacked insulated washer transmission line.

$$\begin{aligned}
V &= \frac{\frac{\partial I}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{R}\right)} + \frac{\frac{\partial^2 I}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)} \\
I &= \frac{\frac{\partial V}{\partial x}}{\frac{\partial}{\partial x}\left(\frac{1}{G}\right)} + \frac{\frac{\partial^2 V}{\partial x \partial t}}{\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}
\end{aligned}$$

We can simplify appearances by defining

$$
\begin{aligned}
a &= \frac{1}{\frac{\partial}{\partial x}\left(\frac{1}{R}\right)} \\
b &= \frac{1}{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)} \\
c &= \frac{1}{\frac{\partial}{\partial x}\left(\frac{1}{G}\right)} \\
d &= \frac{1}{\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}
\end{aligned}
$$

The Telegrapher's Equations for the stacked insulated washer transmission line become

$$
\begin{aligned}
V &= a\frac{\partial I}{\partial x} + b\frac{\partial^2 I}{\partial x \partial t} \\
I &= c\frac{\partial V}{\partial x} + d\frac{\partial^2 V}{\partial x \partial t}
\end{aligned}
$$

We now decouple V and I.

$$
\begin{aligned}
V &= ac\frac{\partial^2 V}{\partial x^2} + (ad+bc)\frac{\partial^3 V}{\partial x^2 \partial t} + bd\frac{\partial^4 V}{\partial x^2 \partial t^2} \\
I &= ac\frac{\partial^2 I}{\partial x^2} + (ad+bc)\frac{\partial^3 I}{\partial x^2 \partial t} + bd\frac{\partial^4 I}{\partial x^2 \partial t^2}
\end{aligned}
$$

We can now use separation of variables. I'll work the voltage equation. The same naturally applies to the current equation.

$$
\begin{aligned}
V(x,t) &= X(x)T(t) \\
X(x)T(t) &= acX''(x)T(t) + (ad+bc)X''(x)T'(t) + bdX''(x)T''(t) \\
\frac{X(x)}{X''(x)} &= \frac{acT(t) + (ad+bc)T'(t) + bdT''(t)}{T(t)} = K
\end{aligned}
$$

The separation constant K has units of area and is proportional to wavelength squared.

For positive values of K, the X generic solution is

$$X(x) = A\exp\left(\frac{x}{\sqrt{K}}\right) + B\exp\left(\frac{-x}{\sqrt{K}}\right)$$

where A and B are generic amplitudes set by initial or boundary conditions.

For negative values of K, the X generic solution is

$$
\begin{aligned}
X(x) &= A\cos\left(\frac{x}{\sqrt{-K}}\right) + B\sin\left(\frac{x}{\sqrt{-K}}\right) \\
&= A\cos(kx) + B\sin(kx)
\end{aligned}
$$

where A and B are again generic amplitudes set by initial or boundary conditions, and are not necessarily the same as the A and B values from the exponential form above.

For this case of $K < 0$, we have wavelength $\lambda = 2\pi\sqrt{-K}$ and wavenumber $k = 1/\sqrt{(-K)}$.

The time equation becomes

$$bdT''(t) + (ad + bc)T'(t) + (ac - K)T(t) = 0$$

This has complex exponential solutions

$$T(t) = C\exp(s_1 t) + D\exp(s_2 t)$$

where

$$s_{1,2} = \frac{-(ad + bc) \pm \sqrt{(ad + bc)^2 + 4bd(K - ac)}}{2bd}$$

In general form, for the damped oscillatory solution, we have

$$T(t) = [C\cos\omega t + D\sin\omega t]\exp(-t/\tau)$$

where

$$\begin{aligned} \tau &= \frac{2bd}{ad + bc} \\ \omega &= \frac{\sqrt{-(ad + bc)^2 - 4bd(K - ac)}}{2bd} \end{aligned}$$

Our overall generic solution (per value of K) becomes

$$\begin{aligned} V(x, t, K) &= (A\cos(kx) + B\sin(kx))(C\cos\omega t + D\sin\omega t)\exp(-t/\tau) \\ &= V_0\cos(kx + \phi_0)\cos(\omega t + \theta_0)\exp(-t/\tau) \end{aligned}$$

In the last line above, the harmonics terms have been combined with a phase shift for the degree of freedom, and the overall scale factors merged into a generic voltage amplitude.

2.2.1 Group and Phase Velocity for the Stacked Insulated Washer Transmission Line

For the high frequency case, where resistance and conductance are negligible, $a, c = 0$, we have

$$V(x, t) = V_0\cos\left(\frac{x}{\sqrt{-K}} + \phi_0\right)\cos\left(t\sqrt{\frac{-K}{bd}} + \theta_0\right)$$

We identify

$$
\begin{aligned}
k &= \frac{1}{\sqrt{-K}} \\
\omega &= \sqrt{\frac{-K}{bd}} = \frac{1}{k}\sqrt{\frac{1}{bd}} \\
\lambda &= 2\pi\sqrt{-K} \\
f &= \frac{1}{2\pi}\sqrt{\frac{-K}{bd}}
\end{aligned}
$$

We find the interesting result that the phase and group velocities are of equal but opposite polarities. (This may be due to coordinate choices in my setup, as transmission lines are bidirectional.)

$$
\begin{aligned}
v_p &= \frac{\omega}{k} = \lambda f = \frac{-K}{\sqrt{bd}} = \frac{1}{k^2}\sqrt{\frac{1}{bd}} \\
v_g &= \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left(\frac{1}{k}\sqrt{\frac{1}{bd}}\right) = -\frac{1}{k^2}\sqrt{\frac{1}{bd}}
\end{aligned}
$$

The velocities are frequency dependent. This is dispersion. Given $\omega = 2\pi f$, we have

$$
\begin{aligned}
v_p &= \omega^2\sqrt{bd} \\
&= (2\pi f)^2 \frac{1}{\sqrt{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}}
\end{aligned}
$$

2.3 Stacked Washer Transmission Line Summary

- High Pass Filter Dominant Characteristic
- Dispersion. High frequencies travel faster than slow frequencies.
- Velocity of Propagation. $v_p = (2\pi f)^2 / \sqrt{\frac{\partial}{\partial x}\left(\frac{1}{L}\right)\frac{\partial}{\partial x}\left(\frac{1}{C}\right)}$

3 Tesla Coil Transmission Lines

The Tesla coil is a single layer solenoid, where turn to turn capacitive effects modify the standard transmission line model.

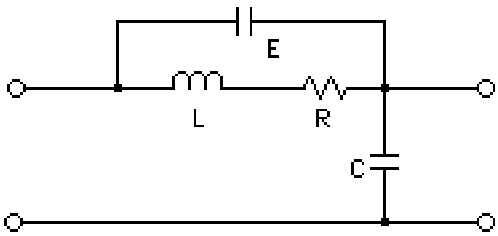

Figure 3 - Tesla Coil Transmission Line

The turn to turn capacitance is modelled as an elastance $E = 1/C$ to simplify the math.

3.1 Differential Equations for the Tesla Coil Transmission Line

The voltage across the inductance is

$$\begin{aligned} V &= L\frac{\partial I_L}{\partial t} + RI_L \\ \frac{\partial V}{\partial x} &= L'\frac{\partial I_L}{\partial t} + R'I_L \end{aligned}$$

The voltage across the elastance E is

$$\begin{aligned} V &= \frac{1}{C}Q = EQ \\ \frac{\partial V}{\partial x} &= E'Q \\ \frac{\partial^2 V}{\partial x \partial t} &= E'I_E \end{aligned}$$

The current through the elastance E is

$$I_E = \frac{1}{E'}\frac{\partial^2 V}{\partial x \partial t}$$

The current loss through the shunting capacitor is found by

$$\begin{aligned} Q &= CV \\ \frac{\partial Q}{\partial x} &= C'V \\ \frac{\partial I}{\partial x} &= C'\frac{\partial V}{\partial t} \end{aligned}$$

The total current is $I = I_L + I_E$. We can use this to separate out voltage and current equations.

$$\begin{aligned} \frac{\partial V}{\partial x} &= L'\frac{\partial I_L}{\partial t} + R'I_L \\ \frac{\partial V}{\partial x} + L'\frac{\partial I_E}{\partial t} + R'I_E &= L'\frac{\partial I_L}{\partial t} + L'\frac{\partial I_E}{\partial t} + R'I_L + R'I_E \\ \frac{\partial V}{\partial x} + L'\frac{\partial I_E}{\partial t} + R'I_E &= L'\frac{\partial I}{\partial t} + R'I \end{aligned}$$

Substituting for I_E on the left hand side, we have

$$\begin{aligned} \frac{\partial V}{\partial x} + L'\frac{\partial I_E}{\partial t} + R'I_E &= L'\frac{\partial I}{\partial t} + R'I \\ \frac{\partial V}{\partial x} + \frac{L'}{E'}\frac{\partial^3 V}{\partial x \partial t^2} + \frac{R'}{E'}\frac{\partial^2 V}{\partial x \partial t} &= L'\frac{\partial I}{\partial t} + R'I \end{aligned}$$

Take the gradient with respect to x on both sides.

$$\frac{\partial^2 V}{\partial x^2} + \frac{L'}{E'}\frac{\partial^4 V}{\partial x^2 \partial t^2} + \frac{R'}{E'}\frac{\partial^3 V}{\partial x^2 \partial t} = L'\frac{\partial^2 I}{\partial x \partial t} + R'\frac{\partial I}{\partial x}$$

Now, substitute for $\partial I/\partial x$ on the right hand side

$$\frac{\partial^2 V}{\partial x^2} + \frac{L'}{E'}\frac{\partial^4 V}{\partial x^2 \partial t^2} + \frac{R'}{E'}\frac{\partial^3 V}{\partial x^2 \partial t} = L'C'\frac{\partial^2 V}{\partial t^2} + R'C'\frac{\partial V}{\partial t}$$

Using separation of variables, $V(x,t) = X(x)T(t)$, we have

$$\begin{aligned}
X''T + \frac{L'}{E'}X''T'' + \frac{R'}{E'}X''T' &= L'C'XT'' + R'C'XT' \\
\frac{X''}{X}\left[T + \frac{L'}{E'}T'' + \frac{R'}{E'}T'\right] &= L'C'T'' + R'C'T' \\
\frac{X''}{X} &= \frac{L'C'T'' + R'C'T'}{\left[\frac{L'}{E'}T'' + \frac{R'}{E'}T' + T\right]} = K
\end{aligned}$$

Our separated differential equations are

$$\begin{aligned}
X'' + X(-K) &= 0 \\
T''\left(L'C' - K\frac{L'}{E'}\right) + T'\left(R'C' - K\frac{R'}{E'}\right) + T(-K) &= 0 \\
T''L' + T'R' + T\left(\frac{-K}{C' - \frac{K}{E'}}\right) &= 0 \\
T'' + T'\frac{R'}{L'} + T\left(\frac{-K}{L'C' - \frac{KL'}{E'}}\right) &= 0
\end{aligned}$$

The spatial equation allows us to identify $K = -(2\pi/\lambda)^2$.

$$X(x) = A\cos\left(\frac{2\pi}{\lambda}x + \phi\right)$$

Substituting for K in the temporal equation, we have

$$T'' + T'\frac{R'}{L'} + T\left(\frac{\left(\frac{2\pi}{\lambda}\right)^2}{L'C' + \left(\frac{2\pi}{\lambda}\right)^2\frac{L'}{E'}}\right) = 0$$

We have the temporal solution

$$T(t) = \exp\left(-t\frac{R'}{2L'}\right)\cos(\omega t + \theta)$$

where

$$\omega = \sqrt{\frac{\left(\frac{2\pi}{\lambda}\right)^2}{L'C' + \left(\frac{2\pi}{\lambda}\right)^2 \frac{L'}{E'}} - \left(\frac{R'}{2L'}\right)^2}$$

We have the generic solution

$$V(x,t,\lambda) = V_0 \exp\left(-t\frac{R'}{2L'}\right) \cos(\omega t + \theta) \cos\left(\frac{2\pi}{\lambda}x + \phi\right)$$

For the case where resistive losses are small, ω approximates as

$$\omega \approx \frac{2\pi/\lambda}{\sqrt{L'C'_{\text{eff}}}}$$

where

$$C'_{\text{eff}} = C' + \left(\frac{2\pi}{\lambda}\right)^2 \frac{1}{E'}$$

At low frequencies (long λ), the coil acts like a conventional transmission line. However, as frequency increases (decreasing λ), the interturn capacitance begins to dominate, and we find a cut-off frequency where the wavelength limits toward zero at finite frequency.

$$\omega_{\text{cutoff}} = \sqrt{\frac{E'}{L'}}$$

The phase velocity for the solenoid is

$$v_p = \frac{\omega}{k} = \lambda f = \frac{\lambda}{2\pi}\sqrt{\frac{\left(\frac{2\pi}{\lambda}\right)^2}{L'C' + \left(\frac{2\pi}{\lambda}\right)^2 \frac{L'}{E'}} - \left(\frac{R'}{2L'}\right)^2}$$

$$v_p = \sqrt{\frac{1}{L'C' + \left(\frac{2\pi}{\lambda}\right)^2 \frac{L'}{E'}} - \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{R'}{2L'}\right)^2}$$

At low frequencies, the phase velocity is constant as with a conventional transmission line. At high frequencies (short λ), the resistive term becomes insignificant, and the effective capacitance increases, dropping the phase velocity. This reflects the increasing impedance of the transmission line as we approach the parallel tank resonance made from the inductive and elastic components.

3.2 Telsa Coil Transmission Line Summary

- Similar to conventional transmission lines at low frequencies.
- Dispersion. High frequencies travel slower than low frequencies.
- Velocity of Propagation. $v_p = \sqrt{\frac{1}{L'C' + \left(\frac{2\pi}{\lambda}\right)^2 \frac{L'}{E'}} - \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{R'}{2L'}\right)^2}$