A Quaternion Toolbox for Four Dimensional Euclidean Spacetime

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September 1, 2008

Introduction

Quaternions occupy an interesting middle ground between vector and tensor algebra. Invented by William Rowan Hamilton, quaternions were to be a divisional algebra covering three dimensional space. Hamilton discovered however, that to extend complex numbers to include three dimensions, he needed four dimensions. The natural fit of quaternion to four dimensional spacetime, especially the divisional algebra character of quaternions, make quaternions a very useful part of the physicists mathematical toolbox.

Spacetime Coordinates

Spacetime consists of three spatial dimensions conjoined with time as a fourth dimension.

Traditionally, we use labels of x, y, z, and t for the coordinates. As we gain in mathematical generality, we use symbols such as x_i as generic coordinates.

The observation that many differential equations combined time rates and spatial gradients helped lead to the recognition of time as a coordinate similar to space.

Relativistic Units

For relativistic units, time will be measured in meters, where 1s = 3.0e8m. In these units, the speed of light is one.

SI Units

In conventional engineering, we use MKS units, where the standard measure of length is the meter, the standard measure of time is the second, and the standard measure of mass is the kilogram and the standard of charge is the Coulomb. Example derived units in this system are

$$
\begin{aligned}
\text{action} \;\rightarrow\; & = 1 \text{ kg m}^2 \text{ s}^{-1} \\
\text{angular momentum} \;\rightarrow\; & = 1 \text{ kg m}^2 \text{ s}^{-1} \\
\text{energy} \;\rightarrow\; 1 \text{ Joule} & = 1 \text{ kg m}^2 \text{ s}^{-2} \\
\text{momentum} \;\rightarrow\; & = 1 \text{ kg m s}^{-1} \\
\text{power} \;\rightarrow\; 1 \text{ Watt} & = 1 \text{ kg m}^2 \text{ s}^{-3} \\
\text{force} \;\rightarrow\; 1 \text{ Newton} & = 1 \text{ kg m s}^{-2} \\
\text{charge} \;\rightarrow\; 1 \text{ Coulomb} & = 6.25 \cdot 10^{18} \text{ electrons} \\
\text{current} \;\rightarrow\; 1 \text{ Amp} & = 1 \text{ Coul s}^{-1} \\
\text{voltage} \;\rightarrow\; 1 \text{ Volt} & = 1 \text{ kg m}^2 \text{ s}^{-2} \text{ Coul}^{-1} \\
\text{magnetic field} \;\rightarrow\; 1 \text{ Tesla} & = 1 \text{ v s m}^{-2} \\
& = 1 \text{ kg s}^{-1} \text{ Coul}^{-1} \\
\text{inductance} \;\rightarrow\; 1 \text{ Henry} & = 1 \text{ kg m}^2 \text{ Coul}^{-2} \\
\text{capacitance} \;\rightarrow\; 1 \text{ Farad} & = 1 \text{ Coul}^2 \text{ s}^2 \text{ m}^{-2} \text{ kg}^{-1}
\end{aligned}
$$

Whimsical Relativistic Units

To make my relativistic units, I substitute 1 s = 299792458.0*m, and change the charge unit from $6.2415 \cdot 10^{18}$ electrons per coulomb to $6.9967 \cdot 10^{15}$ electrons per Q.

$$
\begin{aligned}
1 \text{ s} &= 299792458.0 \text{ m} \\
1 \text{ m/s} &= \text{ velocity} = 3.3356 \cdot 10^{-9} \\
1 \text{ m/s}^2 &= \text{ acceleration} = 1.1127 \cdot 10^{-17} * \text{m}^{-1} \\
1 \text{ kg m/s}^2 &= \text{ force} = 1.1127 \cdot 10^{-17} * \text{ kg} * \text{ m}^{-1} \\
1 \text{ kg m}^2/\text{s}^2 &= \text{ energy} = 1.1127 \cdot 10^{-17} * \text{ kg} \\
1 \text{ kg m}^2/\text{s}^3 &= \text{ power} = 3.7114 \cdot 10^{-26} * \text{ kg} * \text{m}^{-1} \\
1 \text{ kg m/s} &= \text{ momentum} = 3.3356 \cdot 10^{-9} * \text{ kg} \\
1 \text{ kg m}^2/\text{s}^2 &= \text{ torque} = 1.1127 \cdot 10^{-17} * \text{ kg} \\
1 \text{ kg m}^2/\text{s} &= \text{ angular momentum} = 3.3356 \cdot 10^{-9} * \text{ kg} * \text{m} \\
1 \text{ kg m}^2/\text{s} &= \text{ action} = 3.3356 \cdot 10^{-9} * \text{ kg} * \text{m} = 3.3356 \cdot 10^{-9} \text{ axe}
\end{aligned}
$$

$$
\begin{aligned}
1 \text{ Coulomb} &= 6.2415 \cdot 10^{18} \text{ Electrons} = 892.06 \text{ Q} \\
1 \text{ Q} &= 6.9967 \cdot 10^{15} \text{ Electrons} \\
1 \text{ V} &= 9.9255 \cdot 10^{-15} * \text{ kg} * \text{ Q}^{-1} = 9.9255 \cdot 10^{-15} \text{ bolt} \\
1 \text{ Amp} &= 3.7392 \cdot 10^{-12} * \text{m}^{-1} * \text{ Q} = 3.7392 \cdot 10^{-12} \text{ lamp} \\
1 \text{ Tesla} &= 2.9756 \cdot 10^{-6} * \text{ kg} * \text{m}^{-1} * \text{ Q}^{-1} = 2.9756 \cdot 10^{-6} \text{ bee} \\
1 \text{ V/m} &= 9.9255 \cdot 10^{-15} * \text{ kg} * \text{m}^{-1} * \text{ Q}^{-1} = 9.9255 \cdot 10^{-15} \text{ eee} \\
1 \text{ Farad} &= 1.1294 \cdot 10^{11} * \text{ kg}^{-1} * \text{ Q}^2 = 1.1294 \cdot 10^{11} \text{ fred} \\
1 \text{ Henry} &= 7.9577 \cdot 10^{5} * \text{ kg} * \text{m}^2 * \text{ Q}^{-2} = 7.9577 \cdot 10^{5} \text{ hank} \\
\epsilon &= 1.0 \text{ kg}^{-1} * \text{m}^{-1} * \text{ Q}^2 = 1.0 \text{ fred/m} \\
\mu &= 1.0 \text{ kg} * \text{m} * \text{ Q}^{-2} = 1.0 \text{ hank/m} \\
\text{light speed} &= 1.0
\end{aligned}
$$

Comments

These units allow me to ignore terms of c, μ_0 and ϵ_0 when making expressions in electromagnetic theory.

$$
\begin{aligned}
\left(\frac{\phi}{c}, \mathbf{A}\right) &\rightarrow (\phi, \mathbf{A}) \\
\frac{q}{4\pi\epsilon r} &\rightarrow \frac{q}{4\pi r} \\
\frac{\mu q \mathbf{v}}{4\pi r} &\rightarrow \frac{q\mathbf{v}}{4\pi r}
\end{aligned}
$$

A cautionary note concerns Bee and Eee fields. A 1T field appears equivalent to an electric field of $3 \cdot 10^8\ V/m$. This is a bit misleading. While the units of the Bee and Eee fields are the same, in practice, to convert B to E requires a generator term involving $B \times v$ expressions. Since these velocities are now normalized to light speed, the generator term has the high scale factor of the Bee field to compensate for the low velocity scale factor of the velocity term. A rod moving 1 m/s in a 1T fields still produces 1V/m electric field in each case.

Euclidean Four Dimensional Spacetime

Distances Between Events

In four dimensional Cartesian space, the separation between two events P_1 and P_2 is

$$
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (t_1 - t_2)^2}
$$

When dealing with light, the separation in time and space are equal. If light travels a distance $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, the time of flight is also r. (This is due to the unit speed. $dx = vdt \rightarrow dt$ when $v = 1$.)

Consequently, when dealing with electromagnetics, whenever we see radial space separation terms, we can replace this with a spacetime separation with a suitable scale factor.

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (t_1 - t_2)^2} \\
&= \sqrt{r^2 + r^2} \\
d &= r\sqrt{2} \\
r &= d\sqrt{0.5}
\end{aligned}
$$

Four Dimensional Distance Element ds

The differential distance element ds is given by $ds = \sqrt{dt^2 + dx^2 + dy^2 + dz^2}$. This expression is very democratic per the actual basis. However, human experience grants a special significance to time. Consequently, we can recast this in terms of time derivatives, as

$$
\begin{aligned}
ds &= \sqrt{dt^2 + dx^2 + dy^2 + dz^2} \\
&= \sqrt{1 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\
&= dt\sqrt{1 + v^2}
\end{aligned}
$$

Notice that there is no singularity or infinity associated with lightspeed $v = 1$.

Quaternion Four-Vector Notation

William Rowan Hamilton extended complex numbers by defining four basis vectors, $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$, consisting of a real part, and three different imaginary components. The generic quaternion is a four-vector written as a sum of a real number, and three separate imaginary axis numbers.

$$
\tilde{q} = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
$$

where

$$
\begin{aligned}
\mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \\
\mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\
\mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\
\mathbf{ki} &= -\mathbf{ik} = \mathbf{j}
\end{aligned}
$$

Writing the quaternion as all four explicit terms becomes tedious quickly. Interpreting the quaternion as a scalar plus a vector, we can write the generic quaternion using a tilde for the whole quaternion and an overarrow for the vector (spatial) parts.

$$\tilde{q} = q + \vec{Q}$$

Unit Four-Vectors and Direction Cosines

To normalize a four-vector to a unit four-vector, we simply divide by the four-vector magnitude.

$$\begin{aligned}
\tilde{q} &= q + \vec{Q} \\
\tilde{u} &= \frac{q + \vec{Q}}{\sqrt{q^2 + \vec{Q} \cdot \vec{Q}}} \\
\tilde{u} &= \frac{q + Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}}{\sqrt{q^2 + Q_x^2 + Q_y^2 + Q_z^2}}
\end{aligned}$$

The direction cosines associated with t, x, y, and z are

$$\begin{aligned}
\cos(\theta_t) &= \frac{q}{\sqrt{q^2 + \vec{Q} \cdot \vec{Q}}} \\
\cos(\theta_x) &= \frac{Q_x}{\sqrt{q^2 + \vec{Q} \cdot \vec{Q}}} \\
\cos(\theta_y) &= \frac{Q_y}{\sqrt{q^2 + \vec{Q} \cdot \vec{Q}}} \\
\cos(\theta_z) &= \frac{Q_z}{\sqrt{q^2 + \vec{Q} \cdot \vec{Q}}}
\end{aligned}$$

Addition, Subtraction, Scaling and Dot Products

The components of a four vector can be written (t, x, y, z). The magnitude of this vector is $\sqrt{t^2 + x^2 + y^2 + z^2}$.

More commonly, I will use a quaternion notation for four vectors. Define $t + \vec{R}$ as the fourvector above, with time component t and space component $\vec{R} = (x, y, z)$.

Modern physicists will use a generic, tensor notation x_i (covariant) or x^i (contravariant) format. The index i will represent 0-3 or 1-4 depending upon context or author.

We add and subtract fourvectors on a component by component basis.

$$\begin{aligned}
(a, b, c, d) + (t, x, y, z) &= (a + t, b + x, c + y, d + z) \\
(a + \vec{A}) + (b + \vec{B}) &= (a + b) + (\vec{A} + \vec{B}) \\
a_i + b_i &= a_i + b_i \qquad \text{trivial}
\end{aligned}$$

The dot product is the sum of the product of similar coordinates, just as with vectors.

$$\begin{aligned}
(a, b, c, d) \cdot (t, x, y, z) &= at + bx + cy + dz \\
(a + \vec{A}) \cdot (b + \vec{B}) &= ab + (\vec{A} \cdot \vec{B}) \\
a_i \cdot b^i &= a_i b^i \qquad \text{implied sum over high and low indices}
\end{aligned}$$

Scalar products and divisors apply to each component.

$$\begin{aligned}
g * (a, b, c, d) &= (ag, bg, cg, dg) \\
g * (a + \vec{A}) &= ag + \vec{A}g \\
g * (a_i) &= ga_i
\end{aligned}$$

Quaternion Products and Ratios

At the component level the quaternion product is defined as

$$\begin{aligned}
(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) * (e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}) &= (ae - bf - cg - dk) \\
&+ (af + eb + dg - hc)\mathbf{i} \\
&+ (ag + ec + df - bh)\mathbf{j} \\
&+ (ah + ed + bg - dk)\mathbf{k}
\end{aligned}$$

Using the notation $\tilde{q} = q + \vec{Q}$, the quaternion product above can be written as

$$\tilde{a}\tilde{b} = (a+\vec{A})(b+\vec{B}) = (ab - \vec{A}\cdot\vec{B}) + (a\vec{B} + b\vec{A} + \vec{A}\times\vec{B})$$

Quaternion multiplication is not commutative.

$$\begin{aligned}\tilde{a}\tilde{b} &= (a+\vec{A})(b+\vec{B}) = (ab - \vec{A}\cdot\vec{B}) + (a\vec{B} + b\vec{A} + \vec{A}\times\vec{B}) \\ \tilde{b}\tilde{a} &= (b+\vec{B})(a+\vec{A}) = (ab - \vec{A}\cdot\vec{B}) + (a\vec{B} + b\vec{A} - \vec{A}\times\vec{B})\end{aligned}$$

This means that we have a prefactor and postfactor, and that the application formulas using quaternions need to pay attention to the order of multiplication.

The quaternion product has the property that the magnitude of the product is equal to the product of the magnitudes of the individual quaternions. $|\tilde{a}\tilde{b}| = |\tilde{a}||\tilde{b}|$.

This means that the only way a quaternion product can be zero is if one of the two multiplicands were zero.

Given that the magnitude of a quaternion is not zero, we can define the reciprocal of a quaternion.

$$\begin{aligned}\frac{1}{a+\vec{A}} &= \frac{a-\vec{A}}{a^2+\vec{A}\cdot\vec{A}} \quad \text{because} \\ \left(a+\vec{A}\right)\frac{1}{a+\vec{A}} &= \left(a+\vec{A}\right)\frac{a-\vec{A}}{a^2+\vec{A}\cdot\vec{A}} = \frac{a^2+\vec{A}\cdot\vec{A}}{a^2+\vec{A}\cdot\vec{A}} = 1 \quad \text{and} \\ \frac{1}{a+\vec{A}}\left(a+\vec{A}\right) &= \frac{a-\vec{A}}{a^2+\vec{A}\cdot\vec{A}}\left(a+\vec{A}\right) = \frac{a^2+\vec{A}\cdot\vec{A}}{a^2+\vec{A}\cdot\vec{A}} = 1\end{aligned}$$

Although the reciprocal of a quaternion can be unambiguously defined, the non-commutative nature of multiplication means that the order of terms in division is also important.

Expressions like $\tilde{q}/\tilde{r}$ are ambiguous. We must properly specify pre or post division, such as $(1/\tilde{r})\,\tilde{q}$ or $\tilde{q}\,(1/\tilde{r})$.

Just as with complex numbers, we can define a conjugation operator which changes the sign of the imaginary component.

$$\tilde{q} = q + \vec{Q} \tag{1}$$
$$\tilde{q}^* = q - \vec{Q} \tag{2}$$

Just as with complex numbers, quaternions can be characterized as a magnitude and direction.

$$\tilde{a} = a + \vec{A} = a + iA_x + jA_y + kA_z \tag{3}$$
$$|a| = \sqrt{a^2 + A_x^2 + A_y^2 + A_z^2} \tag{4}$$
$$u = (u_t + iu_x + ju_y + ku_z) = \frac{(a + iA_x + jA_y + kA_z)}{\sqrt{a^2 + A_x^2 + A_y^2 + A_z^2}} \tag{5}$$

Just as with complex numbers, quaternions can be parameterized by a magnitude and an angle separation from the real axis.

$$\tilde{q} = q + \vec{Q} \tag{6}$$
$$= \sqrt{q^2 + \vec{Q}\cdot\vec{Q}}\left[\cos\theta + \frac{\vec{Q}}{|\vec{Q}|}\sin\theta\right] \tag{7}$$
$$= A\left[\cos\theta + \vec{U}\sin\theta\right] \tag{8}$$
$$= A\exp\left(\theta\vec{U}\right) \tag{9}$$

where

$$A = \sqrt{q^2 + \vec{Q}\cdot\vec{Q}} \tag{10}$$
$$\vec{U} = \frac{\vec{Q}}{\sqrt{\vec{Q}\cdot\vec{Q}}} \tag{11}$$
$$\theta = \tan^{-1}\frac{q}{\sqrt{\vec{Q}\cdot\vec{Q}}} \tag{12}$$

When two quaternions are multiplied, the product can be thought of as a coupled rotation of components of the first quaternion by the second. One rotation is by the angle θ about time and parallel to $\vec{u}$ axis, while the second rotation is by the angle $-\theta$ of the normal component of $\vec{A}$ around $\vec{u}$.

Let $\tilde{A} = a + \vec{A}$ and $\tilde{B} = \cos\theta + \vec{u}\sin\theta$. The quaternion product of these two terms is

$$\tilde{A}\tilde{B} = (a\cos\theta - (\vec{A}\cdot\vec{u})\sin\theta) + a\sin\theta\vec{u} + \cos\theta\vec{A} + \vec{A}\times\vec{u}\sin\theta \qquad (13)$$

Express $\vec{A}$ in terms of parallel and normal components to $\vec{u}$ via $\vec{A} = \vec{u}\times\left(\vec{A}\times\vec{u}\right) + \vec{u}\left(\vec{A}\cdot\vec{u}\right)$ to obtain

$$\begin{aligned}
\tilde{A}\tilde{B} &= (a\cos\theta - (\vec{A}\cdot\vec{u})\sin\theta) + a\sin\theta\vec{u} + \cos\theta\vec{A} + \vec{A}\times\vec{u}\sin\theta \\
&= \left[\left(a\cos\theta - \left(\vec{A}\cdot\vec{u}\right)\sin\theta\right) + \left(a\sin\theta + \left(\vec{A}\cdot u\right)\right)\vec{u}\cos\theta\right] \\
&\quad + \left[\vec{u}\times\left(\vec{A}\times\vec{u}\right)\cos\theta + \left(\vec{A}\times\vec{u}\right)\sin\theta\right]
\end{aligned}$$

To achieve a simple rotation in space, we can use two quaternion multiplications. First rotate half the desire space amount, and boost half the rotation in the time direction, next change the direction of time dimension rotation by changing the order of multiplication, and finish the spatial rotation while undoing the time rotation. To rotate the vector $\vec{V}$ about axis $\vec{u}$ by angle θ,

$$R(\theta, \vec{u}, \vec{V}) = \exp(-\vec{u}\theta/2)\vec{V}\exp(\vec{u}\theta/2)$$

Divisional Algebras are Found in 1, 2, 4, and 8 Dimensions

Divisional algebras are algebras in 1, 2, 4, and 8 dimensions that allow multiplication and division to be defined. These divisional algebras are usually called out as

1. R - the set of real numbers
2. C - the set of complex numbers
3. Q - the set of quaternions
4. O - the set of octonions

Associated with each algebra is a characteristic square identity.
For real numbers, this is

$$a^2A^2 = (aA)^2$$

For complex numbers, we have the Brahmagupta-Fibonacci relationships

$$\begin{aligned}(a^2+b^2)(A^2+B^2) &= (aA-bB)^2+(bA+aB)^2\\ &= (aA+bB)^2+(bA-aB)^2\end{aligned}$$

For quaternions, we have the Euler four-square relationships

$$\begin{aligned}(a^2+b^2 &+ c^2+d^2)(A^2+B^2+C^2+D^2)\\ &= (aA-bB-cC-dD)^2\\ &+ (aB+bA+cD-dC)^2\\ &+ (aC-bD+cA+dB)^2\\ &+ (aD+bC-cB+dA)^2\end{aligned}$$

For octonions, we have the Degen/Graves/Cayley eight-square identity

$$\begin{aligned}(a^2+b^2 &+ c^2+d^2+e^2+f^2+g^2+h^2)*\\ (A^2 &+ B^2+C^2+D^2+E^2+F^2+G^2+H^2)\\ &= (aA-bB-cC-dD-eE-fF-gG-hH)^2\\ &+ (aB+bA+cD-dC+eF-fE-gH+hG)^2\\ &+ (aC-bD+cA+dB+eG+fH-gE-hF)^2\\ &+ (aD+bC-cB+dA+eH-fG+gF-hE)^2\\ &+ (aE-bF-cG-dH+eA+fB+gC+hD)^2\\ &+ (aF+bE-cH+dG-eB+fA-gD+hC)^2\\ &+ (aG+bH+cE-dF-eC+fD+gA-hB)^2\\ &+ (aH-bG+cF+dE-eD-fC+gB+hA)^2\end{aligned}$$

Clearly, complex numbers have more than one divisional algebra. The top Brahmagupta formula corresponds to conventional complex multiplication, where the angle of the two factors adds for the angle of the product. The second formula, which corresponds to multiplication by the conjugate of the second term, uses differences of angles between the two factors to determine the angle of the product.

Family of Divisional Algebras - Coordinate Inversions and Relabellings

Given multiple formulas that provide divisional algebras in 2 dimensions, the question arises how many different, functional division algebras exist in the 1, 2, 4, and 8 spaces?

The Real Number Family

For the real numbers, we can have arbitrary sign on the product and input terms, and satisy the norm requirement.

$(a)(A) = (+aA)$

$(a)(A) = (-aA)$ both work.

The Complex Number Family

For complex numbers, I find eight working arrangements, based upon different signs for the outputs. Notice that these working configurations always have an odd number of positive signs in the product expressions.

$(a,b)(A,B) = (aA + bB, aB - bA)$

$(a,b)(A,B) = (aA + bB, -aB + bA)$

$(a,b)(A,B) = (aA - bB, aB + bA)$

$(a,b)(A,B) = (-aA + bB, aB + bA)$

$(a,b)(A,B) = (-aA - bB, -aB + bA)$

$(a,b)(A,B) = (-aA - bB, aB - bA)$

$(a,b)(A,B) = (-aA + bB, -aB - bA)$

$(a,b)(A,B) = (aA - bB, -aB - bA)$

In a similar manner, we find 256 algebras for 4 dimensions, and at least 512K divisional algebras for 8 dimensions. These different forms are related by inverting the product term component polarity, and by relabelling of output components.

My opinion is that we can match physics to our default quaternion by suitable sign conventions and the use of the conjugation operator.

For example, to use a left handed product rather than the right hand product, we change the order of multiplication.

$$\begin{aligned}
\tilde{a}\tilde{b} &= (ab - \vec{A}\cdot\vec{B}) + a\vec{B} + b\vec{A} + \vec{A}\times\vec{B} \quad \text{right hand system} \\
\tilde{b}\tilde{a} &= (ab - \vec{A}\cdot\vec{B}) + a\vec{B} + b\vec{A} + \vec{B}\times\vec{A} \quad \text{left hand system}
\end{aligned}$$

To change the sign of the vector term, we can use conjugation or reciprocals. However, it is really nice to know we have many possibilities for divisional basis.

Quaternion Products and Ratios with $d\tilde{r}$ and the Unit Tangent $\tilde{u}$

In four dimensional space,

$$\begin{aligned}
d\tilde{r} &= dt + \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz \\
d\tilde{r} &= (1 + \vec{v})\, dt
\end{aligned}$$

The differential length element ds is (choosing the positive root)

$$\begin{aligned}
ds^2 &= d\tilde{r}\cdot d\tilde{r} = dt^2 + dx^2 + dy^2 + dz^2 \\
ds &= \sqrt{d\tilde{r}\cdot d\tilde{r}} = \sqrt{dt^2 + dx^2 + dy^2 + dz^2} = dt\sqrt{1+v^2}
\end{aligned}$$

The unit tangent is

$$\tilde{u} = \frac{d\tilde{r}}{ds} = \frac{1+\vec{v}}{\sqrt{1+v^2}}$$

A very useful relationship is ratio of a quaternion with the unit vector. This ratio gives the colinear and transverse components of the field in terms of the four-vector unit velocity of the particle.

$$
\begin{aligned}
\frac{1}{\tilde{u}}\tilde{q} &= \tilde{u}^*\tilde{q} \\
&= \left(u - \vec{U}\right)\left(q + \vec{Q}\right) \\
&= \left(uq + \vec{U}\cdot\vec{Q}\right) + u\vec{Q} - q\vec{U} + \vec{Q}\times\vec{U}
\end{aligned}
$$

The time component is the magnitude of the colinear component, while the space portion is the transverse component.

Some Useful Combinations

Some useful combinations of pairs of quaternions are listed below.

$$
\begin{aligned}
\frac{1}{2}\left[\tilde{q}\tilde{p} + \tilde{p}\tilde{q}\right] &= \left(pq - \vec{P}\cdot\vec{Q}\right) + q\vec{P} + p\vec{Q} \\
\frac{1}{2}\left[\tilde{q}\tilde{p} - \tilde{p}\tilde{q}\right] &= \vec{Q}\times\vec{P} \\
\frac{1}{2}\left[\tilde{p}\tilde{q}^* + \tilde{q}\tilde{p}^*\right] &= pq + \vec{P}\cdot\vec{Q} \\
\frac{1}{2}\left[\tilde{p}\tilde{q} + \tilde{q}^*\tilde{p}^*\right] &= pq - \vec{P}\cdot\vec{Q}
\end{aligned}
$$

Some interesting triples of quaternions are listed below. Note $\tilde{u} = u + \vec{U}$ is a unit magnitude quaternion.

$$
\begin{aligned}
\tilde{u}\tilde{a}\tilde{u} &= a\tilde{u}^2 - 2(\vec{A}\cdot\vec{U})\tilde{u} + \vec{A} \\
\tilde{u}^*\tilde{a}\tilde{u} &= a + 2\vec{U}(\vec{A}\cdot\vec{U}) + (u^2 - U^2)\vec{A} + u\vec{A}\times\vec{U}
\end{aligned}
$$

Repeated application of unit vectors has interesting formulas.

$$
\begin{aligned}
\vec{U}\vec{U} &= 0 - \vec{U}\cdot\vec{U} + 0\vec{U} + 0\vec{U} + \vec{U}\times\vec{U} = -1 \\
\vec{U}\tilde{q}\vec{U} &= \vec{U}\left[\left(-\vec{Q}\cdot\vec{U}\right) + q\vec{U} + \vec{Q}\times\vec{U}\right] \\
&= -q + \left(-\vec{Q}\cdot\vec{U}\right)\vec{U} + \vec{U}\times\left(\vec{Q}\times\vec{U}\right) \\
&= -q + \vec{Q} - 2\vec{U}(\vec{Q}\cdot\vec{U})
\end{aligned}
$$

For the special case of two normal unit vectors, we have

$$\vec{U}\vec{N}\vec{U} = \vec{N}$$

A relationship we will find useful later deals with orthogonal relationships between quaternions.

The dot product between two quaternions is defined by

$$\tilde{p} \cdot \tilde{q} = pq + \vec{P} \cdot \vec{Q}$$

Examine the situation where $\tilde{p}\tilde{q} \perp \tilde{p}$. What are the restrictions that this places on $\tilde{q}$?

$$\begin{aligned}
\tilde{p}\tilde{q} &= \left(pq - \vec{P} \cdot \vec{Q}\right) + p\vec{Q} + q\vec{P} + \vec{P} \times \vec{Q} \\
\tilde{p} \cdot (\tilde{p}\tilde{q}) &= p\left(pq - \vec{P} \cdot \vec{Q}\right) + p\vec{P} \cdot \vec{Q} + q\vec{P} \cdot \vec{P} + 0 \\
&= q\left(p^2 + \vec{P} \cdot \vec{P}\right)
\end{aligned}$$

For normal 4 vectors, the dot product is zero, and this requires either $q = 0$ or $|\tilde{p}| = 0$.

We thus see that multiplying a vector times a quaternion results in a normal product, and similarly, that normal quaternions are related by a spatial vector product.

Moving a factor from pre-factor to post-factor, and vice versus is useful at times. Consider the following expression.

$$\tilde{g}\tilde{a} = \tilde{a}\tilde{h}$$

In general, we can find $\tilde{g}$ in terms of $\tilde{a}$ and $\tilde{h}$, as well as find $\tilde{h}$ in terms of $\tilde{a}$ and $\tilde{g}$.

$$\begin{aligned}
\tilde{g}\tilde{a} &= \tilde{a}\tilde{h} \\
\tilde{g} &= \frac{\tilde{a}\tilde{h}\tilde{a}^*}{a^2} \\
\tilde{h} &= \frac{\tilde{a}^*\tilde{g}\tilde{a}}{a^2}
\end{aligned}$$

We see that $\tilde{g}$ is $\tilde{h}$ rotated by twice the angle of $\tilde{a}$.

As an example, $(1 + \vec{v})(\vec{a} + \vec{a} \times \vec{v}) = (\vec{a} - \vec{a} \times \vec{v})(1 + \vec{v})$.

Quaternion Calculus

The quaternion differential is simply the differential of its parts.

$$d\tilde{f} = df + d\vec{F}$$

The special case of four dimensional radial vectors is interesting.

$$d\tilde{r} = d(t + \vec{R}) = dt + d\vec{R} = (1 + \vec{v})dt$$

Derivatives and integrals are subject to potential ambiguities, as multiplication and division are non-commutative. The cross product terms will have a sign change depending upon the choice of pre- or post-multiplication or division. Bearing this in mind, one definition for the quaternion derivative (using post-division) is

$$\begin{aligned}
\frac{d\tilde{f}}{d\tilde{g}} &= d\tilde{f}\frac{1}{d\tilde{g}} \\
&= d\tilde{f}\left(\frac{dg - d\vec{G}}{(dg)^2 + (d\vec{G} \cdot d\vec{G})}\right) \\
&= \frac{df\ dg + d\vec{F} \cdot d\vec{G} - df d\vec{G} + dg d\vec{F} - d\vec{F} \times d\vec{G}}{(dg)^2 + (d\vec{G} \cdot d\vec{G})}
\end{aligned}$$

For the special case where the post-denominator is a radial quaternion, we have the form

$$
\begin{aligned}
\frac{d\tilde{f}}{d\tilde{r}} &= \frac{d\tilde{f}}{dt + d\vec{R}} \\
&= \frac{d\tilde{f}}{(1+\vec{v})\,dt} \\
&= \frac{d\tilde{f}}{dt}\left(\frac{1}{1+\vec{v}}\right) \\
&= \left(\frac{df}{dt} + \frac{d\vec{F}}{dt}\right)\frac{(1-\vec{v})}{1+v^2} \\
&= \left(\frac{df}{dt} + \vec{v}\cdot\frac{d\vec{F}}{dt} - \vec{v}\frac{df}{dt} + \frac{d\vec{F}}{dt} + \vec{v}\times\frac{d\vec{F}}{dt}\right)\left(\frac{1}{1+v^2}\right)
\end{aligned}
$$

Integration is also straightforward, as we can convert radial terms to time integrals fairly easily.

$$
\begin{aligned}
\int \tilde{f}d\tilde{r} &= \int (f+\vec{F})d(t+\vec{R}) \\
&= \int (f+\vec{F})(1+\vec{v})dt \\
&= \int \left(f - \vec{v}\cdot\vec{F} + \vec{F} + f\vec{v} + \vec{F}\times\vec{v}\right)dt
\end{aligned}
$$

We also have the nice relationship

$$
\frac{d}{dt}\int \tilde{f}d\tilde{r} = f - \vec{v}\cdot\vec{F} + \vec{F} + f\vec{v} + \vec{F}\times\vec{v}
$$

Quaternion Partial Differential Operator

The last topic is the quaternion partial differential operator $\partial/\partial t + \vec{\nabla}$. Applying this operator to a quaternion $a + \vec{A}$ yields an expression similar to potential theory expressions.

$$
\left(\frac{\partial}{\partial t} + \vec{\nabla}\right)\left(a + \vec{A}\right) = \left(\frac{\partial a}{\partial t} - \vec{\nabla}\cdot\vec{A}\right) + \vec{\nabla}a + \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\times\vec{A}
$$

Applying this operator twice yields an expression reminiscent of the vector wave equation.

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \vec{\nabla}\right)^2 \left(a + \vec{A}\right) &= \left(\frac{\partial^2 a}{\partial t^2} - \nabla^2 a - 2\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right)\right) \\ &+ \left(\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} + 2\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right) + 2\frac{\partial}{\partial t}\left(\vec{\nabla} a\right)\right)\end{aligned}$$

What does this operator do? Let's start by looking at the derivative of some simple expressions. ($\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$)

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(t + \vec{R}) &= -2 \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(t - \vec{R}) &= 4 \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(t + \vec{R})(a + \vec{A}) &= -2a - 2\vec{A} \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(a + \vec{A})(t + \vec{R}) &= -2a + 2\vec{A} \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(t + \vec{R})^2 &= -4t \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)(t^2 + R^2) &= 2(t + \vec{R}) \\ \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)\frac{1}{t + \vec{R}} &= \frac{2}{t^2 + R^2}\end{aligned}$$

We see that any radial function's quaternion derivative folds back into a scalar.

Now, let's take the four dimensional divergence of this differential operator. The four dimensional divergence is given by

$$\tilde{\nabla}_4 \cdot \tilde{f} = \frac{\partial}{\partial t} f_t + \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$

Applying this to the quaternion partial derivative yields

$$
\begin{aligned}
\tilde{\nabla}_4 \cdot \left(\frac{\partial}{\partial t} + \vec{\nabla}\right)\left(\phi + \vec{A}\right) &= \tilde{\nabla}_4 \cdot \left[\frac{\partial \phi}{\partial t} - \vec{\nabla}\cdot\vec{A}, \vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\times\vec{A}\right] \\
&= \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi
\end{aligned}
$$

We note the significant loss of information associated with the four dimensional divergence.

Comparison with Electromagnetics

Using voltage and the magnetic vector potential, we can write the quaternion gradient of the potential four-vector as

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \vec{\nabla}\right)\left(\phi + \vec{A}\right) &= \left(\frac{\partial \phi}{\partial t} - \vec{\nabla}\cdot\vec{A}\right) + \left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right) + \vec{\nabla}\times\vec{A} \\
&= \left(\frac{\partial \phi}{\partial t} - \vec{\nabla}\cdot\vec{A}\right) - \vec{E} + \vec{B}
\end{aligned}
$$

The time component, which corresponds to power over charge, looks like a modified Lorentz guage criterion, while the space component is like a square root, vector form of free energy density in space.

Quaternion Frenet-Serret Formula for Trajectories using Scalar Curvatures

Using quaternion notation, the left hand coordinate Frenet equations can be written as

$$\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} \\
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\frac{d\tilde{w}}{ds} &= -\gamma\tilde{b}
\end{aligned}$$

It is useful to calculate the curvature, torsion and lift in terms of three dimensional velocities, accelerations, and related time derivatives.

Define four dimensional position $\tilde{r}$ as

$$\tilde{r} = t + \vec{r} = t + x\mathbf{i} + \mathbf{yj} + \mathbf{zk}$$

Define three dimensional position $\vec{r}$ as

$$\vec{r} = x\mathbf{i} + \mathbf{yj} + \mathbf{zk}$$

Define velocity $\vec{v}$ as

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Define acceleration $\vec{a}$ as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Define jerk $\vec{j}$ as

$$\vec{j} = \frac{d\vec{a}}{dt} = \frac{d^3\vec{r}}{dt^3}$$

Define yank $\vec{y}$ as

$$\vec{y} = \frac{d\vec{j}}{dt} = \frac{d^4\vec{r}}{dt^4}$$

The differential pathlength in spacetime is defined by

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2$$

Expressing ds in terms of our familiar 3D space derivatives, we have

$$ds = \sqrt{dx^2 + dy^2 + dz^2 + dt^2} = dt\sqrt{1 + v^2}$$

The unit tangent in 4-space is given (using quaternion notation) by

$$\tilde{u} = \frac{d\tilde{r}}{ds} = \frac{1 + \vec{v}}{\sqrt{1 + v^2}}$$

The normal is given by

$$\tilde{n} = \frac{-(\vec{v} \cdot \vec{a}) + \vec{a} + \vec{v} \times (\vec{a} \times \vec{v})}{\sqrt{(1 + v^2)\,[\vec{a} + \vec{a} \times \vec{v}] \cdot [\vec{a} + \vec{a} \times \vec{v}]}}$$

Curvature is given by

$$\kappa = \sqrt{\frac{[\vec{a} + \vec{a} \times \vec{v}] \cdot [\vec{a} + \vec{a} \times \vec{v}]}{(1 + v^2)^3}}$$

Straightforward application of the rules for derivatives can lead to correct, but obtuse formulas for the torsion and lift. A more informative method of calculating torsion and lift in terms of our common three dimensional velocity, acceleration and jerk will be to notice what appears to be a quaternion product, and develop quaternion based formulas for our items of interest.

We notice that the normal is the quaternion product of the tangent and a three dimensional unit vector.

$$\tilde{n} \;=\; \left(\frac{1 + \vec{v}}{\sqrt{1 + v^2}}\right)\left(\frac{\vec{a} + \vec{a} \times \vec{v}}{\sqrt{(\vec{a} + \vec{a} \times \vec{v}) \cdot (\vec{a} + \vec{a} \times \vec{v})}}\right)$$

$$= \tilde{u}\frac{\vec{G}}{|\vec{G}|} = \tilde{u}\vec{g}$$

We earlier noted that the ratio of two orthogonal quaternions is a vector. In the case of the basis vectors for a Frenet system, it follows that each of the basis vectors is related to the other basis vectors by means of a unit vector multiplier. We also note that for terms such as $\vec{g}\vec{h}$ or $\vec{g}\vec{h}\vec{L}$ to be a pure vector, $\vec{g}$, $\vec{h}$ and $\vec{L}$ must be orthogonal. We thus write the following definitions.

$$\begin{aligned}
\tilde{n} &= \tilde{u}\vec{g} \\
\tilde{b} &= \tilde{n}\vec{h} = \tilde{u}\vec{g}\vec{h} \\
\tilde{w} &= \tilde{b}\vec{L} = \tilde{u}\vec{g}\vec{h}\vec{L}
\end{aligned}$$

We also developed that $\tilde{a}\vec{u}\vec{u} = -\tilde{a}$ and $\vec{u}\vec{g}\vec{u} = \vec{g} - 2\vec{u}\,(\vec{g}\cdot\vec{u}) = \vec{g}$ when $\vec{u}$ is a unit vector perpendicular to $\vec{g}$.

Let us now calculate $d\tilde{n}/ds$, and compare the derived form with the Frenet formula definition.

$$\begin{aligned}
\frac{d\tilde{n}}{ds} &= \frac{d}{ds}\left(\tilde{u}\vec{g}\right) \\
&= \frac{d\tilde{u}}{ds}\vec{g} + \tilde{u}\frac{d\vec{t}}{ds} \\
&= \kappa\tilde{n}\vec{g} + \tilde{u}\frac{d\vec{g}}{ds} \\
&= \kappa\tilde{u}\vec{g}\vec{g} + \tilde{u}\frac{d\vec{g}}{ds} \\
&= -\kappa\tilde{u} + \tilde{u}\frac{d\vec{g}}{ds}
\end{aligned}$$

Comparing this with the Frenet formula, we identify

$$\tau\tilde{b} = \tilde{u}\frac{d\vec{g}}{ds}$$

Further, since $\tilde{b}$ and $\tilde{u}$ are both unit magnitude, we have

$$|\tau| = \left|\frac{d\vec{g}}{ds}\right|$$

Next, evaluate

$$\begin{aligned}
\frac{d\tilde{w}}{ds} &= \frac{d\tilde{b}\vec{L}}{ds} \\
&= \frac{d\tilde{b}}{ds}\vec{L} + \tilde{b}\frac{d\vec{L}}{ds} \\
&= \left(\gamma\tilde{w} - \tau\tilde{n}\right)\vec{L} + \tilde{b}\frac{d\vec{L}}{ds} \\
&= \gamma\tilde{b}\vec{L}\vec{L} - \tau\tilde{n}\vec{L} + \tilde{b}\frac{d\vec{L}}{ds} \\
&= -\gamma\tilde{b} + \tilde{b}\frac{d\vec{L}}{ds} - \tau\tilde{n}\vec{L}
\end{aligned}$$

Comparison to the matching Frenet equation leads to the conclusion that

$$\tilde{b}\frac{d\vec{L}}{ds} = \tau\tilde{n}\vec{L}$$

Since $\tilde{n}, \tilde{b}$ and $\vec{L}$ are all unit magnitude, we also have

$$|\tau| = \left|\frac{d\vec{L}}{ds}\right|$$

We thus see that $|\vec{g}|$ may be equal to $|\vec{L}|$. Using $\vec{g} = -\vec{L}$ gives a left hand coordinate system, and leads to the relationship $\tilde{w} = -\tilde{b}\vec{g} = -\tilde{u}\vec{g}\vec{h}\vec{g} = -\tilde{u}\vec{h}$, if $\vec{g} \perp \vec{h}$, as proves to be, later.

Now we evaluate

$$\begin{aligned}
\frac{d\tilde{b}}{ds} &= \frac{d\tilde{n}\vec{h}}{ds} = \frac{d\tilde{n}}{ds}\vec{h} + \tilde{n}\frac{d\vec{h}}{ds} \\
&= \left(\tau\tilde{b} - \kappa\tilde{u}\right)\vec{h} + \tilde{n}\frac{d\vec{h}}{ds} \\
&= \tau\tilde{n}\vec{h}\vec{h} + \kappa\tilde{w} + \tilde{n}\frac{d\vec{h}}{ds} \\
&= -\tau\tilde{n} + \left(\tilde{n}\frac{d\vec{h}}{ds} + \kappa\tilde{w}\right)
\end{aligned}$$

Comparing this with the Frenet formula, we see that

$$\gamma\tilde{w} = \kappa\tilde{w} + \tilde{n}\frac{d\vec{h}}{ds}$$

Now we want to see what relationship exists between the vectors $\vec{g}$ and $\vec{h}$. We start by looking at the Frenet formula

$$\begin{aligned}
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
&= \tau\left(\tilde{u}\vec{g}\vec{h}\right) - \kappa\tilde{u} \\
\frac{d\tilde{u}\vec{g}}{ds} &= \tau\left(\tilde{u}\vec{g}\vec{h}\right) - \kappa\tilde{u} \\
\frac{d\tilde{u}}{ds}\vec{g} + \tilde{u}\frac{d\vec{g}}{ds} &= \tau\left(\tilde{u}\vec{g}\vec{h}\right) - \kappa\tilde{u}
\end{aligned}$$

Postmultiply both sides by $\vec{g}$ and obtain

$$\begin{aligned}
\frac{d\tilde{u}}{ds}\vec{g}\vec{g} + \tilde{u}\frac{d\vec{g}}{ds}\vec{g} &= \tau\tilde{u}\vec{g}\vec{h}\vec{g} - \kappa\tilde{u}\vec{g} \\
-\frac{d\tilde{u}}{ds} + \tilde{u}\frac{d\vec{g}}{ds}\vec{g} &= \tau\tilde{u}\vec{g}\vec{h}\vec{g} - \kappa\tilde{u}\vec{g} \\
-\kappa\tilde{n} + \tilde{u}\frac{d\vec{g}}{ds}\vec{g} &= \tau\tilde{u}\vec{g}\vec{h}\vec{g} - \kappa\tilde{n} \\
\tilde{u}\frac{d\vec{g}}{ds}\vec{g} &= \tau\tilde{u}\vec{g}\vec{h}\vec{g} \\
\tilde{u}\frac{d\vec{g}}{ds}\vec{g} &= \tau\tilde{u}\left(\vec{h} - 2\left(\vec{g}\cdot\vec{h}\right)\right) \\
\frac{d\vec{g}}{ds}\vec{g} &= \tau\left(\vec{h} - 2\left(\vec{g}\cdot\vec{h}\right)\right)
\end{aligned}$$

For the case of these unit vectors, $d\vec{g}/ds$ is perpendicular to $\vec{g}$, thus if

$$\vec{h} = \frac{1}{\tau}\frac{d\vec{g}}{ds} \times \vec{g}$$

then we would have $\vec{h} \perp \vec{g}$, and

$$\frac{d\vec{g}}{ds}\vec{g} = \tau\vec{h}$$

We see that $\vec{g}$, $\vec{g}\vec{h}$ and $\vec{h}$ make up a normal triad.

Now let's write the expressions for our basis vectors, curvature, torsion and lift.

From earlier,

$$\tilde{u} = \frac{1+\vec{v}}{\sqrt{1+v^2}}$$

$$\vec{g} = \frac{\vec{a}+\vec{a}\times\vec{v}}{\sqrt{a^2+(\vec{a}\times\vec{v})^2}}$$

$$\begin{aligned}\tilde{n} &= \tilde{u}\vec{g} = \frac{1+\vec{v}}{\sqrt{1+v^2}}\frac{\vec{a}+\vec{a}\times\vec{v}}{\sqrt{a^2+(\vec{a}\times\vec{v})^2}} \\ &= \frac{-\vec{v}\cdot\vec{a}+\vec{a}\,(1+v^2)-\vec{v}\,(\vec{a}\cdot\vec{v})}{\sqrt{1+v^2}\sqrt{a^2+(\vec{a}\times\vec{v})^2}}\end{aligned}$$

$$\begin{aligned}\frac{d\vec{g}}{ds} &= \frac{\vec{j}+\vec{j}\times\vec{v}}{\sqrt{1+v^2}\sqrt{a^2+(\vec{a}\times\vec{v})^2}} + \vec{g}\frac{d}{ds}(...) \\ \frac{d\vec{g}}{ds}\times\vec{g} &= \frac{\vec{j}+\vec{j}\times\vec{v}}{\sqrt{1+v^2}\sqrt{a^2+(\vec{a}\times\vec{v})^2}}\times\vec{g}+0=\tau\vec{h}\end{aligned}$$

The unit vector $\vec{h}$ is

$$\vec{h} = \frac{\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})}{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}$$

$$
\begin{aligned}
\tilde{b} &= \tilde{u}\vec{g}\vec{h} \\
&= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\frac{[\vec{a}+\vec{a}\times\vec{v}]\times\left[\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right]}{\sqrt{a^2+(\vec{a}\times\vec{v})^2}\left|\left(\vec{j}+\vec{j}\times v\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{w} &= -\tilde{b}\vec{g} = -\tilde{u}\vec{h} \\
&= -\left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\left[\frac{\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})}{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}\right]
\end{aligned}
$$

We note that the direction of the unit vectors are determined by velocity, acceleration and jerk.

From earlier, the curvature is

$$
\begin{aligned}
\kappa &= \frac{1}{\tilde{n}}\frac{d\tilde{u}}{ds} \\
|\kappa| &= \left|\frac{d\tilde{u}}{ds}\right| = \sqrt{\frac{a^2+(\vec{a}\times\vec{v})^2}{(1+v^2)^3}}
\end{aligned}
$$

The torsion is

$$
\begin{aligned}
\tau &= \frac{1}{\tilde{b}}\tilde{u}\frac{d\vec{g}}{ds} \\
|\tau| &= \left|\frac{d\vec{g}}{ds}\right| \\
&= \frac{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}{\left(a^2+(\vec{a}\times\vec{v})^2\right)\sqrt{1+v^2}} \\
&= \frac{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}{\kappa^2\left(1+v^2\right)^{7/2}}
\end{aligned}
$$

The expression for lift is a bit tricky to derive. We should first define some intermediate symbols and expressions.

Define 'jerk' $\vec{j} = d\vec{a}/dt$, being the time derivate of acceleration.
Define 'yank' $\vec{y} = d\vec{j}/dt$, being the time derivative of jerk.
Define

$$\begin{aligned}
\vec{G} &= \vec{a} + \vec{a} \times \vec{v} \\
\vec{g} &= \frac{\vec{G}}{|\vec{G}|} \\
\vec{G}' &= \vec{j} + \vec{j} \times \vec{v} \\
\vec{G}'' &= \vec{y} + \vec{y} \times \vec{v} + \vec{j} \times \vec{a}
\end{aligned}$$

Derivatives of unit vectors are always normal to that unit vector, and the formula has a simple form.

$$\begin{aligned}
\frac{d}{ds}\vec{g} &= \frac{d}{ds}\frac{\vec{G}}{\sqrt{\vec{G} \cdot \vec{G}}} \\
&= \frac{\vec{G} \times \left(\vec{G}' \times \vec{G}\right)}{|\vec{G}|^3\sqrt{1+v^2}} \\
&= \frac{\vec{g} \times \left(\vec{G}' \times \vec{G}\right)}{|\vec{G}|^2\sqrt{1+v^2}}
\end{aligned}$$

Define

$$\begin{aligned}
\vec{H} &= \vec{G}' \times \vec{G} \\
&= \left(\vec{j} + \vec{j} \times \vec{v}\right) \times (\vec{a} + \vec{a} \times \vec{v}) \\
&= \left(\vec{j} \times \vec{a}\right) + \left(\vec{j} \times \vec{a}\right) \times \vec{v} + \vec{v}\left(\vec{j}, \vec{a}, \vec{v}\right)
\end{aligned}$$

where $\left(\vec{j}, \vec{a}, \vec{v}\right)$ is the triple vector product $\vec{j} \cdot (\vec{a} \times \vec{v})$.

Notice that $\vec{H}' = \vec{G}'' \times \vec{G}$. Thus

$$\vec{H}' \times \vec{H} = -\vec{G}\left(\vec{G}', \vec{G}'', \vec{G}\right)$$

Also note that

$$\vec{H} \cdot \vec{H} = \left[\left(\vec{j} \times \vec{a}\right)^2 + \left(\vec{j}, \vec{a}, \vec{v}\right)^2\right]\left(1 + v^2\right)$$

So, to get the expression for lift, we look at

$$\begin{aligned}
\frac{d\tilde{b}}{ds} &= \frac{d\left(\tilde{n}\vec{h}\right)}{ds} = \frac{d\tilde{n}}{ds}\vec{h} + \tilde{n}\frac{d\vec{h}}{ds} \\
\gamma\tilde{w} - \tau\tilde{n} &= \left(\tau\tilde{b} - \kappa\tilde{u}\right)\vec{h} + \tilde{n}\frac{d\vec{h}}{ds} \\
\gamma\tilde{w} &= \kappa\tilde{w} + \tilde{n}\frac{d\vec{h}}{ds} \\
\gamma\tilde{w} &= \kappa\tilde{w} + \tilde{n}\vec{h}\left(\frac{\vec{H}' \times H}{\sqrt{1+v^2}|\vec{H}|^2}\right) \\
\gamma\tilde{w} &= \kappa\tilde{w} + \tilde{n}\vec{h}\left(\frac{-\vec{G}\left(\vec{G}', \vec{G}'', \vec{G}\right)}{\sqrt{1+v^2}|\vec{H}|^2}\right) \\
\gamma\tilde{w} &= \kappa\tilde{w} + \tilde{n}\tilde{g}\vec{h}\left(\frac{|\vec{G}|\left(\vec{G}', \vec{G}'', \vec{G}\right)}{\sqrt{1+v^2}|\vec{H}|^2}\right) \\
\gamma &= \kappa + \left(\frac{|\vec{G}|\left(\vec{G}', \vec{G}'', \vec{G}\right)}{|\vec{H}|^2\sqrt{1+v^2}}\right)
\end{aligned}$$

Expand $\vec{G}''$, and separate terms in $\vec{y}$,

$$\begin{aligned}
\gamma &= \kappa + \frac{|\vec{G}|}{|\vec{H}|^2}\frac{\left(\vec{G}', \left(\vec{j} \times \vec{a}\right), \vec{G}\right)}{\sqrt{1+v^2}} + \frac{|\vec{G}|}{|\vec{H}|^2}\frac{\left(\vec{G}', \left(\vec{y} + \vec{y} \times \vec{v}\right), \vec{G}\right)}{\sqrt{1+v^2}} \\
&= \kappa + \frac{|\vec{G}|}{\left(1+v^2\right)^{3/2}}\frac{\left(\vec{G}', \left(\vec{j} \times \vec{a}\right), \vec{G}\right)}{\left[\left(\vec{j} \times \vec{a}\right)^2 + \left(\vec{j}, \vec{a}, \vec{v}\right)\right]} + \frac{|\vec{G}|}{|\vec{H}|^2}\frac{\left(\vec{G}', \left(\vec{y} + \vec{y} \times \vec{v}\right), \vec{G}\right)}{\sqrt{1+v^2}} \\
&= \kappa - \kappa + \frac{|\vec{G}|}{|\vec{H}|^2}\frac{\left(\vec{G}', \left(\vec{y} + \vec{y} \times \vec{v}\right), \vec{G}\right)}{\sqrt{1+v^2}} \\
&= \frac{|\vec{G}|}{|\vec{H}|^2}\frac{\left(\vec{G}', \left(\vec{y} + \vec{y} \times \vec{v}\right), \vec{G}\right)}{\sqrt{1+v^2}}
\end{aligned}$$

This yields

$$\begin{aligned}
\gamma &= \kappa \frac{\vec{G}', (\vec{y} + \vec{y} \times \vec{v}), \vec{G}}{\left[\left(\vec{j} \times \vec{a} \right)^2 + \left(\vec{j}, \vec{a}, \vec{v} \right)^2 \right]} \\
&= \kappa \frac{\vec{H} \cdot (\vec{y} + \vec{y} \times \vec{v})}{\vec{H} \cdot \left(\vec{j} \times \vec{a} \right)}
\end{aligned}$$

The most important fact to be noticed from this expression is that the lift is a function of velocity, acceleration, jerk and yank. Hence, to define any curve in 4-D space time, we must be prepared to include expressions of velocity, acceleration, jerk and yank.

Summary

$$\begin{aligned}
\vec{G} &= \vec{a} + \vec{a} \times \vec{v} \\
\vec{H} &= \left(\vec{j} + \vec{j} \times \vec{v} \right) \times (\vec{a} + \vec{a} \times \vec{v}) \\
\kappa &= \frac{|\vec{a} + \vec{a} \times \vec{v}|}{(1 + v^2)^{3/2}} = \frac{|\vec{G}|}{(1 + v^2)^{3/2}} \\
\tau &= \frac{\left| \left(\vec{j} + \vec{j} \times \vec{v} \right) \times (\vec{a} + \vec{a} \times \vec{v}) \right|}{\kappa^2 (1 + v^2)^{7/2}} = \frac{|\vec{H}|}{\kappa^2 (1 + v^2)^{7/2}} \\
\gamma &= \kappa \frac{\vec{H} \cdot (\vec{y} + \vec{y} \times \vec{v})}{\vec{H} \cdot \left(\vec{j} \times \vec{a} \right)}
\end{aligned}$$

Unit Vectors

$$
\begin{aligned}
\tilde{u} &= \frac{1+\vec{v}}{\sqrt{1+v^2}} \\
\tilde{n} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right) \frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
&= \tilde{u}\frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
\tilde{b} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right) \frac{\left[(\vec{a}+\vec{a}\times\vec{v})\times\left(\vec{j}+\vec{j}\times\vec{v}\right)\right]\times[\vec{a}+\vec{a}\times\vec{v}]}{\sqrt{a^2+(\vec{a}\times\vec{v})^2}\left|\left(\vec{j}+\vec{j}\times v\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|} \\
&= \tilde{w}\frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
\tilde{w} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\left[\frac{(\vec{a}+\vec{a}\times\vec{v})\times\left(\vec{j}+\vec{j}\times\vec{v}\right)}{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}\right]
\end{aligned}
$$

Quaternion Frenet-Serret Formulas using Vector Curvatures

These equations specify the trajectory of a particle along a curve using the curvature κ, torsion τ, and lift γ as scalar items. These scalars suffer from sign ambiguity. Think of a sine wave curve in two dimensions. From zero to 180 degrees, the curvature is negative, but from 180 to 360, the curvature is positive. We can seek a vector relationship for curvature similar to the right hand rule from electromagnetics.

Given that the ratio of two orthogonal quaternions is a vector, let's look at the Frenet formulas deriving vector quantities which correspond to the curvature, torsion and lift scalars.

Curvature Vector, Tangent and Normal

Start with curvature.

$$\begin{aligned}
\frac{d\tilde{u}}{ds} &= \kappa\tilde{n} = \tilde{u}\frac{\vec{a} + \vec{a}\times\vec{v}}{\left(1+v^2\right)^{3/2}} = \tilde{u}\vec{\kappa} \\
\vec{\kappa} &= \frac{1}{\tilde{u}}\frac{d\tilde{u}}{ds} = \frac{\vec{a} + \vec{a}\times\vec{v}}{\left(1+v^2\right)^{3/2}}
\end{aligned}$$

Torsion Vector, Normal and Binormal

Likewise for torsion, form the normal ratio

$$\begin{aligned}
\frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} \\
\frac{1}{\tilde{n}}\frac{d\tilde{n}}{ds} &= \tau\frac{1}{\tilde{n}}\tilde{b} - \kappa\frac{1}{\tilde{n}}\tilde{u} = \vec{\tau} - \kappa\frac{1}{\tilde{n}}\tilde{u} \\
\vec{\tau} &= \frac{1}{\tilde{n}}\frac{d\tilde{n}}{ds} + \kappa\frac{1}{\tilde{n}}\tilde{u} \\
&= \frac{1}{\tilde{n}}\frac{d(\tilde{u}\vec{g})}{ds} + \kappa\frac{1}{\tilde{n}}\tilde{u} \\
&= \frac{1}{\tilde{n}}\tilde{u}\frac{d\vec{g}}{ds} + \frac{1}{\tilde{n}}\frac{d\tilde{u}}{ds}\vec{g} + \kappa\frac{1}{\tilde{n}}\tilde{u} \\
&= (-\vec{g})\frac{d\vec{g}}{ds} + \frac{1}{\tilde{n}}(\kappa\tilde{n})\vec{g} + \kappa(-\vec{g}) \\
\vec{\tau} &= (-\vec{g})\frac{d\vec{g}}{ds} \\
\vec{\tau} &= (-\vec{g})\frac{(\vec{a} + (\vec{a}\times\vec{v}))\times\left(\left(\vec{j} + \left(\vec{j}\times\vec{v}\right)\right)\times(\vec{a} + (\vec{a}\times\vec{v}))\right)}{\sqrt{1+v^2}\left(a^2 + (\vec{a}\times\vec{v})^2\right)^{3/2}} \\
\vec{\tau} &= (-\vec{g})(\vec{g})\frac{\left(\vec{j} + \left(\vec{j}\times\vec{v}\right)\right)\times(\vec{a} + (\vec{a}\times\vec{v}))}{\sqrt{1+v^2}\left(a^2 + (\vec{a}\times\vec{v})^2\right)} \\
\vec{\tau} &= \frac{\left(\vec{j} + \left(\vec{j}\times\vec{v}\right)\right)\times(\vec{a} + (\vec{a}\times\vec{v}))}{\sqrt{1+v^2}\left(a^2 + (\vec{a}\times\vec{v})^2\right)}
\end{aligned}$$

We note that

$$\frac{1}{\tilde{n}}\frac{d\tilde{n}}{ds} = \vec{\tau} + \vec{\kappa}$$

Lift Vector, Binormal and Trinormal

Similarly, for the lift, we start with

$$
\begin{aligned}
\frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} \\
\frac{1}{\tilde{b}}\frac{d\tilde{b}}{ds} &= \frac{1}{\tilde{b}}\gamma\tilde{w} - \tau\frac{1}{\tilde{b}}\tilde{n} \\
&= \vec{\gamma} + \vec{\tau} \\
\vec{\gamma} &= -\vec{\tau} + \frac{1}{\tilde{b}}\frac{d\tilde{b}}{ds} \\
&= -\vec{\tau} + \frac{1}{\tilde{b}}\frac{d(\tilde{n}\vec{h})}{ds} \\
&= -\vec{\tau} + \frac{1}{\tilde{b}}\frac{d\tilde{n}}{ds}\vec{h} + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\tau} + \frac{1}{\tilde{b}}\left[\tilde{n}\left(\vec{\tau} + \vec{\kappa}\right)\right]\vec{h} + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\tau} + \vec{h}^{*}\tilde{n}^{*}\left[\tilde{n}\left(\vec{\tau} + \vec{\kappa}\right)\right]\vec{h} + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\tau} - \vec{h}\left[\left(\vec{\tau} + \vec{\kappa}\right)\right]\vec{h} + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\tau} - \left[-2\left(\vec{h}\cdot\left(\vec{\tau} + \vec{\kappa}\right)\right)\vec{h} + \left(\vec{\tau} + \vec{\kappa}\right)\right] + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\tau} - \left[-2\vec{\tau} + \left(\vec{\tau} + \vec{\kappa}\right)\right] + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\kappa} + \frac{1}{\tilde{b}}\tilde{n}\frac{d\vec{h}}{ds} \\
&= -\vec{\kappa} - \vec{h}\frac{d\vec{h}}{ds} \\
&= -\vec{\kappa} - \vec{h}\left(\vec{h}\left(\frac{\vec{H}'\times\vec{H}}{\sqrt{1+v^2}H^2}\right)\right) \\
&= -\vec{\kappa} + \left(\frac{\vec{H}'\times\vec{H}}{\sqrt{1+v^2}H^2}\right) \\
\vec{\gamma} &= -\vec{\kappa} - \left(\frac{\vec{G}\left(\vec{G}', \vec{G}'', \vec{G}\right)}{\sqrt{1+v^2}H^2}\right) \\
&= -\left(\frac{\vec{G}\left(\vec{G}', \left(\vec{y} + \vec{y}\times\vec{v}\right), \vec{G}\right)}{\sqrt{1+v^2}H^2}\right) \\
&= \left(\frac{\vec{G}\left(\vec{G}', \vec{G}, \left(\vec{y} + \vec{y}\times\vec{v}\right)\right)}{\sqrt{1+v^2}H^2}\right)
\end{aligned}
$$

Notice that $\vec{\gamma}$ is parallel to κ.
Using these vector definitions, the Frenet equations become

$$\begin{aligned}
\frac{d\tilde{u}}{ds} &= \tilde{u}\vec{\kappa} \\
\frac{d\tilde{n}}{ds} &= \tilde{n}(\vec{\tau}+\vec{\kappa}) \\
\frac{d\tilde{b}}{ds} &= \tilde{b}(\vec{\gamma}+\vec{\tau}) \\
\frac{d\tilde{w}}{ds} &= \tilde{w}\vec{\gamma}
\end{aligned}$$

Vector Curvature Frenet-Serret Formulas

Dynamical Equations

$$\begin{aligned}
\frac{d\tilde{u}}{ds} &= \tilde{u}\vec{\kappa} \\
\frac{d\tilde{n}}{ds} &= \tilde{n}(\vec{\tau}+\vec{\kappa}) \\
\frac{d\tilde{b}}{ds} &= \tilde{b}(\vec{\gamma}+\vec{\tau}) \\
\frac{d\tilde{w}}{ds} &= \tilde{w}\vec{\gamma}
\end{aligned}$$

Unit Basis

$$
\begin{aligned}
\tilde{u} &= \frac{1+\vec{v}}{\sqrt{1+v^2}} \\
\tilde{n} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
&= \tilde{u}\frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
\tilde{b} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\frac{\left[(\vec{a}+\vec{a}\times\vec{v})\times\left(\vec{j}+\vec{j}\times\vec{v}\right)\right]\times[\vec{a}+\vec{a}\times\vec{v}]}{\sqrt{a^2+(\vec{a}\times\vec{v})^2}\left|\left(\vec{j}+\vec{j}\times v\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|} \\
&= \tilde{w}\frac{(\vec{a}+\vec{a}\times\vec{v})}{\sqrt{(\vec{a})^2+(\vec{a}\times\vec{v})^2}} \\
\tilde{w} &= \left(\frac{1+\vec{v}}{\sqrt{1+v^2}}\right)\left[\frac{(\vec{a}+\vec{a}\times\vec{v})\times\left(\vec{j}+\vec{j}\times\vec{v}\right)}{\left|\left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})\right|}\right]
\end{aligned}
$$

Vector Curvatures

$$
\begin{aligned}
\vec{\kappa} &= \frac{1}{\tilde{u}}\frac{d\tilde{u}}{ds} = \frac{\vec{a}+\vec{a}\times\vec{v}}{(1+v^2)^{3/2}} \\
\vec{\tau} &= \frac{\left(\vec{j}+\left(\vec{j}\times\vec{v}\right)\right)\times(\vec{a}+(\vec{a}\times\vec{v}))}{\sqrt{1+v^2}\left(a^2+(\vec{a}\times\vec{v})^2\right)} \\
\vec{\gamma} &= \vec{G}\frac{\left(\vec{H}\cdot(\vec{y}+\vec{y}\times\vec{v})\right)}{\sqrt{1+v^2}H^2}
\end{aligned}
$$

where

$$
\begin{aligned}
\vec{G} &= \vec{a}+\vec{a}\times\vec{v} \\
\vec{H} &= \left(\vec{j}+\vec{j}\times\vec{v}\right)\times(\vec{a}+\vec{a}\times\vec{v})
\end{aligned}
$$