Quaternion Curvatures

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Abstract

Four dimensional curves can be parameterized along pathlength by three scalars using the Frenet-Serret formulas. Using quaternion notation, these three scalars can be generalized into curvature, torsion and lift vectors. I provide formulas for the curvature, torsion and lift using quaternions parameterized by an arbitrary monotonically increasing parameter, such as fourspace pathlength, or proper time, which allows for curves oscillating in the time direction without derivative infinities as would be incurred in parameterization by time.

Motivation

Spinor representations of the electron can be interpreted as a four-dimensional oscillator. I want to form a quasi-classical model of a four-dimensional electron as oscillating in time and space, capturing relativistic behavior and correctly modelling spin effects.

Quaternion Definitions and Notations

Quaternions were created as a way to provide division of two vectors. When two vectors are colinear, their ratio is a scalar, but when the two vectors are normal, their ratio was seen to be a third normal vector. Hamilton reconciled these two by creating a four-dimensional extension of vectors, composed of a scalar and a vector component. Hamilton extended the complex numbers to have three complex basis i, j, k, one for each spatial dimension x, y, and z. His formulas are often summarized as $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. I will ordinarily not write out the spatial vectors as $xi + yj + zk$, as I am trying to reduce clutter. Instead, I will show the spatial components as a vector with an overbar, like $\vec{R} = xi + yj + zk$.

When I write a generic quaternion, I will use tilde over the symbol, as in $\tilde{q}$. When I want to emphasize the composite nature of a quaternion as a number plus a vector, I will use a notation like $\tilde{q} = q + \vec{Q}$ or sometimes $\tilde{q} = (q, \vec{Q})$, where the scalar part (time part) of the quaternion is the scalar q, and the vector part is $\vec{Q}$.

The quaternion multiplication law for two quaternions is

$$\begin{aligned} \tilde{q}\tilde{r} &= (q + \vec{Q})(r + \vec{R}) \\ &= (qr - \vec{Q} \cdot \vec{R}) + q\vec{R} + r\vec{Q} + \vec{Q} \times \vec{R} \end{aligned}$$

This multiplication law is associative, but non-commutative. $\tilde{a}(\tilde{b}\tilde{c}) = (\tilde{a}\tilde{b})\tilde{c}$, but $\tilde{a}\tilde{b} \neq \tilde{b}\tilde{a}$ in the general case. In addition to the quaternion multiplication above, we have the standard dot product between two quaternions, indicated by a centered dot.

$$\tilde{q} \cdot \tilde{r} = qr + \vec{Q} \cdot \vec{R}$$

This is used as part of the magnitude of quaternions formula, $q^2 = \tilde{q} \cdot \tilde{q} = \tilde{q}\tilde{q}^*$.

The inverse of a quaternion is

$$\frac{1}{\tilde{q}} = \frac{\tilde{q}^*}{\tilde{q}\tilde{q}^*} = \frac{q - \vec{Q}}{q^2 + \vec{Q} \cdot \vec{Q}}$$

As multiplication is non-commutative, so division. Pre-division is, in general, not equal to post-division.

$$\frac{1}{\tilde{q}}\tilde{r} \neq \tilde{r}\frac{1}{\tilde{q}}$$

I will explicitly show the order of multiplications and inversions in the formulas which follow.

Parameterization by θ

Ordinarily, we parameterize engineering calculations by time, and deal with positions, velocities, accelerations and so forth. This becomes problematic when dealing with four dimensional quantities, such as time travellers or particles which change direction in time as part of their self-history. Their associated velocities, accelerations and higher order derivatives suffer infinities as $dt = 0$.

To avoid these infinities, we need to use another parameterization. When dealing with purely geometric consideration of curves, a natural parameterization is the path length s. For my four dimensional curves, measuring time in meters, $ds^2 = dt^2 + dx^2 + dy^2 + dz^2$. This ds is a good candidate for the proper time, or self-history time of the particle or time traveller.

However, for greatest generality, I will develop formulas here parameterized by a generic parameter θ. In some cases, this could be time. In other cases, it could be s. In other cases, it might be an angular measure along an orbit, as done in a simulation. By leaving this parameter generic, I will avoid an interesting flaw in formulation, where setting the parameter, say to time, implies even time steps, and thus kills any information on changing rates of time from showing in the derived formulas.

As a consequence, we will see chain rule type forms in our expressions for derivatives. Let $\tilde{q} = \tilde{q}(\theta)$. Our generic position vector in four-space, specifying a location and time, is

$$\tilde{r} = t(\theta) + \vec{R}(\theta)$$

Taking the derivative yields

$$\begin{aligned} \tilde{dr} &= dt + d\vec{R} \\ &= \left(\frac{dt}{d\theta} + \frac{d\vec{R}}{d\theta} \right) d\theta \end{aligned}$$

The terms in the parentheses are a generic velocity, which I will denotes as $\tilde{v} = v_t(\theta) + \vec{V}(\theta)$. Taking the magnitude of the last equation above, we have our formula for ds,

$$ds = \sqrt{v_t^2 + \vec{V} \cdot \vec{V}} d\theta = v d\theta$$

When I am doing derivatives with respect to pathlength ds, these factors $1/v$ will be common.

Orthogonal Quaternions Have a Vector Ratio

With complex numbers, multiplying by i rotates the original number 90°. Likewise for quaternions, multiplication by a pure vector creates a perpendicular quaternion.

$$\begin{aligned}\tilde{q}\vec{R} &= (-\vec{Q}\cdot\vec{R}) + q\vec{R} + \vec{Q}\times\vec{R} \\ \tilde{q}\cdot(\tilde{q}\vec{R}) &= (q+\vec{Q})\cdot((-\vec{Q}\cdot\vec{R}) + q\vec{R} + \vec{Q}\times\vec{R}) \\ &= q(-\vec{Q}\cdot\vec{R}) + \vec{Q}\cdot(q\vec{R}) + \vec{Q}\cdot(\vec{Q}\times\vec{R}) \\ &= 0\end{aligned}$$

Due to the divisional alegebra of quaternions, we can invert this relationship, and say that any two perpendicular quaternions are related by a pure vector factor. We will use this fact to greatly simplify our work in the following sections.

Frenet-Serret Equations in Quaternion Format

The Frenet-Serret equations describe a four-dimensional trajectory using curvature, torsion, and lift as a function of distance. We will present the traditional Frenet-Serret equations, then re-express the same equations, using vector quantities for the three curvatures, and showing the relationship between the orthogonal basis and these curvature vectors.

The traditional Frenet-Serret equations in left hand format are on the left, while the vectorized quaternion format is on the right.

$$\begin{aligned}\frac{d\tilde{r}}{ds} &= \tilde{u} \\ \frac{d\tilde{u}}{ds} &= \kappa\tilde{n} = \tilde{u}\vec{\kappa} \\ \frac{d\tilde{n}}{ds} &= \tau\tilde{b} - \kappa\tilde{u} = \tilde{n}(\vec{\tau}+\vec{\kappa}) \\ \frac{d\tilde{b}}{ds} &= \gamma\tilde{w} - \tau\tilde{n} = \tilde{b}(\vec{\gamma}+\vec{\tau}) \\ \frac{d\tilde{w}}{ds} &= -\gamma\tilde{b} = \tilde{w}\vec{\gamma}\end{aligned}$$

On the second line above, we know $\tilde{u}$ and $\tilde{n}$ are orthogonal, and therefore related by a vector factor. We assign that direction to $\vec{\kappa}$. We have the truism

$$\tilde{n} = \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)$$

If we quaternion post-multiply this expression by $\vec{\kappa}$, we find

$$\begin{aligned}
\tilde{n}\tilde{\kappa} &= \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)\vec{\kappa} \\
&= \tilde{u}\left(\frac{\vec{\kappa}\vec{\kappa}}{\kappa}\right) \\
&= \tilde{u}\left(\frac{-\kappa^2}{\kappa}\right) \\
&= -\kappa\tilde{u}
\end{aligned}$$

We now see from the expression for $d\tilde{n}/ds$ above, we can define

$$\begin{aligned}
\tau\tilde{b} &= \tilde{n}\vec{\tau} \quad \text{or} \\
\tilde{b} &= \tilde{n}\left(\frac{\vec{\tau}}{\tau}\right) = \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)\left(\frac{\vec{\tau}}{\tau}\right)
\end{aligned}$$

In a similar fashion, we find

$$\tilde{w} = \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right) = \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)\left(\frac{\vec{\tau}}{\tau}\right)\left(\frac{\vec{\gamma}}{\gamma}\right)$$

We have an interesting observation about the directions of these curvature vectors. To keep $\tilde{b}$ normal to $\tilde{u}$, $\vec{\kappa}$ must be perpendicular to $\vec{\tau}$. This means $\vec{\kappa}\vec{\tau} = \vec{\kappa} \times \vec{\tau}$. To keep $\tilde{w}$ orthogonal to the other basis vectors, we cannot have $\vec{\gamma}$ parallel with $\vec{\kappa} \times \vec{\tau}$, as that would scalarize the triple product in the expression for $\tilde{w}$ in terms of $\tilde{u}$. Instead, we find that $\vec{\gamma}$ must be parallel (or anti-parallel) with $\vec{\kappa}$. This, in turns, simplifies the expression for $\tilde{w}$.

$$\begin{aligned}
\tilde{n} &= \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right) \\
\tilde{b} &= \tilde{n}\left(\frac{\vec{\tau}}{\tau}\right) = \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{w} &= \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right) = \tilde{u}\left(\frac{\vec{\tau}}{\tau}\right)
\end{aligned}$$

Clean summary of quaternionic Frenet-Serret equations:

$$
\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \tilde{u}\vec{\kappa} \\
\frac{d\tilde{n}}{ds} &= \tilde{n}(\vec{\tau}+\vec{\kappa}) \\
\frac{d\tilde{b}}{ds} &= \tilde{b}(\vec{\gamma}+\vec{\tau}) \\
\frac{d\tilde{w}}{ds} &= \tilde{w}\vec{\gamma} \\
\tilde{n} &= \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right) \\
\tilde{b} &= \tilde{n}\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{w} &= \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right)
\end{aligned}
$$

Curvature, Torsion and Lift

Define derivatives with respect to a parameter θ, which is not necessarily time or distance.

$$
\begin{aligned}
\frac{d\tilde{r}}{d\theta} &= \tilde{v} \\
\frac{d\tilde{v}}{d\theta} &= \tilde{a} \\
\frac{d\tilde{a}}{d\theta} &= \tilde{j} \\
\frac{d\tilde{j}}{d\theta} &= \tilde{y}
\end{aligned}
$$

While I will call the derivatives $\tilde{v}$ velocity, $\tilde{a}$ acceleration, $\tilde{j}$ jerk, and $\tilde{y}$ yank, these are four-vectors, not three-vectors, and the derivative is definitely not limited to derivatives with respect to time.

Our elemental distance unit is related to velocity and θ, as

$$\begin{aligned} d\tilde{r} \cdot d\tilde{r} &= (ds)^2 \\ ds &= \sqrt{d\tilde{r} \cdot d\tilde{r}} \\ &= \sqrt{\frac{d\tilde{r}}{d\theta} \cdot \frac{d\tilde{r}}{d\theta}} d\theta \\ ds &= v d\theta \end{aligned}$$

We have a formula of convenience to present.

$$\begin{aligned} \tilde{v} \cdot \tilde{v} &= v^2 \\ 2\tilde{v} \cdot \tilde{a} &= 2v\frac{dv}{d\theta} \\ \frac{dv}{d\theta} &= \frac{\tilde{v} \cdot \tilde{a}}{v} \end{aligned}$$

In a similar fashion

$$\begin{aligned} \frac{da}{d\theta} &= \frac{\tilde{a} \cdot \tilde{j}}{a} \\ \frac{dj}{d\theta} &= \frac{\tilde{j} \cdot \tilde{y}}{j} \end{aligned}$$

Find the Curvature Vector

$$\begin{aligned} \frac{d\tilde{u}}{ds} &= \frac{d\tilde{u}}{v d\theta} = \frac{1}{v}\frac{d}{d\theta}\left(\frac{\tilde{v}}{v}\right) \\ &= \frac{1}{v}\left[\frac{\tilde{a}}{v} - \tilde{v}\frac{1}{v^2}\frac{dv}{d\theta}\right] \\ &= \frac{1}{v}\left[\frac{\tilde{a}}{v} - \frac{\tilde{v}}{v^2}\frac{\tilde{v} \cdot \tilde{a}}{v}\right] \\ &= \frac{\tilde{a}v^2 - \tilde{v}(\tilde{a} \cdot \tilde{v})}{v^4} \\ &= \frac{\tilde{v}}{v}\left(\frac{\tilde{v}^*\tilde{a} - \tilde{a} \cdot \tilde{v}}{v^3}\right) \\ &= \tilde{u}\vec{\kappa} \end{aligned}$$

From this, we see that

$$\vec{\kappa} = \frac{\tilde{v}^*\tilde{a} - \tilde{a} \cdot \tilde{v}}{v^3}$$

To get the magnitude of the curvature, it is easier to work with the definition of the normal.

$$
\begin{aligned}
\frac{d\tilde{u}}{ds} &= \frac{\tilde{a}v^2 - \tilde{v}(\tilde{a}\cdot\tilde{v})}{v^4} = \tilde{n}\kappa \\
\kappa^2 &= \frac{\tilde{a}v^2 - \tilde{v}(\tilde{a}\cdot\tilde{v})}{v^4} \cdot \frac{\tilde{a}v^2 - \tilde{v}(\tilde{a}\cdot\tilde{v})}{v^4} \\
&= \frac{a^2v^4 + v^2(\tilde{a}\cdot\tilde{v})^2 - 2(\tilde{a}\cdot\tilde{v})^2v^2}{v^8} \\
&= \frac{a^2v^4 - v^2(\tilde{a}\cdot\tilde{v})^2}{v^8} \\
&= \frac{a^2v^2 - (\tilde{a}\cdot\tilde{v})^2}{v^6} \\
\kappa &= \frac{\sqrt{a^2v^2 - (\tilde{a}\cdot\tilde{v})^2}}{v^3}
\end{aligned}
$$

Find the Torsion Vector

Begin with the definition of the normal.

$$
\begin{aligned}
\tilde{n} &= \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right) \\
\frac{d\tilde{n}}{ds} &= \frac{d\tilde{u}}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right) + \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right) \\
&= \kappa\tilde{n}\left(\frac{\vec{\kappa}}{\kappa}\right) + \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right) \\
&= \tilde{n}\vec{\kappa} + \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right)
\end{aligned}
$$

Comparing with the Frenet-Serret equation, we see that

$$
\begin{aligned}
\frac{d\tilde{n}}{ds} &= \tilde{n}\vec{\kappa} + \tilde{n}\vec{\tau} \\
&= \tilde{n}\vec{\kappa} + \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right) \\
\tilde{n}\vec{\tau} &= \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right)
\end{aligned}
$$

Substituting for $\tilde{n}$, and cleaning up, we find

$$
\begin{aligned}
\tilde{n}\vec{\tau} &= \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right) \\
\tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right)\vec{\tau} &= \tilde{u}\left(\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)\right) \\
\vec{\tau} &= \left(\frac{\kappa}{\vec{\kappa}}\right)\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right) \\
&= \left(\frac{\kappa}{\vec{\kappa}}\right)\left(\frac{1}{\kappa}\frac{d\vec{\kappa}}{ds} - \frac{\vec{\kappa}}{\kappa^2}\frac{d\kappa}{ds}\right) \\
&= \left(\frac{\kappa}{\vec{\kappa}}\right)\left(\frac{1}{\kappa}\frac{d\vec{\kappa}}{ds} - \frac{\vec{\kappa}}{\kappa^2}\frac{1}{\kappa}\left(\vec{\kappa}\cdot\frac{d\vec{\kappa}}{ds}\right)\right) \\
&= \left(\frac{\kappa}{\vec{\kappa}}\right)\frac{\vec{\kappa}\times\left(\frac{d\vec{\kappa}}{ds}\times\vec{\kappa}\right)}{\kappa^3} \\
\vec{\tau} &= \frac{1}{\kappa^2}\frac{d\vec{\kappa}}{ds}\times\vec{\kappa}
\end{aligned}
$$

We can write $d\vec{\kappa}/ds$ fairly easily.

$$
\begin{aligned}
\vec{\kappa} &= \frac{\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a}}{v^3} \\
\frac{d\vec{\kappa}}{ds} &= \frac{1}{v}\frac{d}{d\theta}\left(\frac{\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a}}{v^3}\right) \\
&= \frac{1}{v}\left[\frac{\tilde{a}^*\tilde{a} + \tilde{v}^*\tilde{j} - \tilde{a}\cdot\tilde{a} - \tilde{v}\cdot\tilde{j}}{v^3} + (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\frac{d}{d\theta}\left(\frac{1}{v^3}\right)\right] \\
&= \frac{1}{v}\left[\frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{v^3} + (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\left(\frac{-3}{v^4}\frac{dv}{d\theta}\right)\right] \\
&= \frac{1}{v}\left[\frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{v^3} + (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\left(\frac{-3(\tilde{v}\cdot\tilde{a})}{v^5}\right)\right] \\
&= \frac{1}{v}\left[\frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{v^3} + \vec{\kappa}\left(\frac{-3(\tilde{v}\cdot\tilde{a})}{v^2}\right)\right]
\end{aligned}
$$

We now do our cross product.

$$
\frac{d\vec{\kappa}}{ds}\times\vec{\kappa} = \frac{1}{v}\left[\frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{v^3}\right]\times\frac{\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a}}{v^3}
$$

Divide by κ^2 to get

$$\vec{\tau} = \frac{(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})}{v(v^2a^2 - (\tilde{v}\cdot\tilde{a})^2)}$$

We have two factorization forms that may or may not be interesting. First, showing a unit vector postfactor,

$$\vec{\tau} = \frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{v\sqrt{v^2a^2 - (\tilde{v}\cdot\tilde{a})^2}} \times \frac{\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a}}{\sqrt{v^2a^2 - (\tilde{v}\cdot\tilde{a})^2}}$$

The second factorization clearly calls out a factor of $\vec{\kappa}$.

$$\vec{\tau} = \left(\frac{\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}}{a^2 - (\vec{v}\cdot\vec{a}/v)^2}\right) \times \vec{\kappa}$$

Lift Vector

The lift vector surprised me a bit. As mentioned earlier, the lift vector is colinear with the curvature vector. Let's prove this assertion now.

We earlier showed

$$\vec{\tau} = -\frac{\vec{\kappa}}{\kappa}\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right)$$

We now examine the trinormal base equation.

$$\begin{aligned}
\tilde{w} &= \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right) \\
\frac{d\tilde{w}}{ds} &= \frac{d\tilde{b}}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right) + \tilde{b}\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right) = \tilde{w}\vec{\gamma} \\
&= \tilde{b}\left(\vec{\tau} + \vec{\gamma}\right)\left(\frac{\vec{\gamma}}{\gamma}\right) + \tilde{b}\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right) = \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right)\vec{\gamma}
\end{aligned}$$

This implies

$$\begin{aligned}
(\vec{\tau} + \vec{\gamma})\left(\frac{\vec{\gamma}}{\gamma}\right) + \frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right) &= \left(\frac{\vec{\gamma}}{\gamma}\right)\vec{\gamma} \quad \text{or} \\
\vec{\tau}\left(\frac{\vec{\gamma}}{\gamma}\right) &= -\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right)
\end{aligned}$$

Post-multiply by $\vec{\gamma}/\gamma$, and find

$$\begin{aligned}
\vec{\tau}\left(\frac{\vec{\gamma}}{\gamma}\right)\left(\frac{\vec{\gamma}}{\gamma}\right) &= -\left(\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right)\right)\left(\frac{\vec{\gamma}}{\gamma}\right) \\
-\vec{\tau} &= -\left(\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right)\right)\left(\frac{\vec{\gamma}}{\gamma}\right) \\
\vec{\tau} &= \left(\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right)\right)\left(\frac{\vec{\gamma}}{\gamma}\right) \\
\vec{\tau} &= -\left(\frac{\vec{\gamma}}{\gamma}\right)\frac{d}{ds}\left(\frac{\vec{\gamma}}{\gamma}\right) \\
&= -\frac{\vec{\kappa}}{\kappa}\frac{d}{ds}\left(\frac{\vec{\kappa}}{\kappa}\right) \quad \text{from above}
\end{aligned}$$

Comparing the two formulas for $\vec{\tau}$, we are forced to conclude that the two unit vectors $\vec{\kappa}/\kappa$ and $\vec{\gamma}/\gamma$ are colinear, either parallel or antiparallel, but colinear.

We now develop our first formula for $\vec{\gamma}$. We begin with our definition for $\tilde{b}$.

$$\begin{aligned}
\tilde{b} &= \tilde{n}\left(\frac{\vec{\tau}}{\tau}\right) \\
\frac{d\tilde{b}}{ds} &= \frac{d\tilde{n}}{ds}\left(\frac{\vec{\tau}}{\tau}\right) + \tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) = \tilde{b}\left(\vec{\tau}+\vec{\gamma}\right) \\
&= \left(\tau\tilde{b}-\kappa\tilde{u}\right)\left(\frac{\vec{\tau}}{\tau}\right) + \tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) = \tilde{b}\vec{\tau}+\tilde{b}\vec{\gamma}
\end{aligned}$$

This implies

$$\begin{aligned}
\tilde{b}\vec{\tau}-\kappa\tilde{u}\left(\frac{\vec{\tau}}{\tau}\right)+\tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) &= \tilde{b}\vec{\tau}+\tilde{b}\vec{\gamma} \\
\tilde{b}\vec{\tau}+\tilde{b}\vec{\gamma} &= \tilde{b}\vec{\tau}-\kappa\tilde{u}\left(\frac{\vec{\tau}}{\tau}\right)+\tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{b}\vec{\gamma} &= -\kappa\tilde{u}\left(\frac{\vec{\tau}}{\tau}\right)+\tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right)
\end{aligned}$$

We now clear the $\tilde{b}$ forefactor

$$\begin{aligned}
\tilde{b}\vec{\gamma} &= -\kappa\tilde{u}\left(\frac{\vec{\tau}}{\tau}\right) + \tilde{n}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{b}\vec{\gamma} &= -\kappa\tilde{w} + \tilde{n}\frac{\vec{\tau}}{\tau}\frac{\tau}{\vec{\tau}}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{b}\vec{\gamma} &= -\kappa\tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right) + \tilde{b}\frac{\tau}{\vec{\tau}}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) \\
\vec{\gamma} &= -\kappa\left(\frac{\vec{\gamma}}{\gamma}\right) - \frac{\vec{\tau}}{\tau}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right)
\end{aligned}$$

Earlier I commented that the lift and curvature vectors could be anti-parallel. We can see such a case here when torsion is constant. Transpose the curvature term to the left, and take the constant factor γ.

$$\begin{aligned}
\vec{\gamma} + \kappa\left(\frac{\vec{\gamma}}{\gamma}\right) &= -\frac{\vec{\tau}}{\tau}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) \\
\vec{\gamma}\left(1 + \frac{\kappa}{\gamma}\right) &= -\frac{\vec{\tau}}{\tau}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right)
\end{aligned}$$

This implies $\kappa = -\gamma$ when $\vec{\tau}$ is constant. Thus we see antiparallel alignment in this instance. We now substitute $\vec{\gamma}/\gamma = \vec{\kappa}/\kappa$, and write our first form for $\vec{\gamma}$

$$\vec{\gamma} = -\vec{\kappa} - \left(\frac{\vec{\tau}}{\tau}\right)\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right)$$

We now need to evaluate the derivative on the right hand side. Primes are derivatives with respect to s.

$$\begin{aligned}
\vec{\tau} &= \frac{\vec{\kappa}' \times \vec{\kappa}}{\kappa^2} \\
\frac{\vec{\tau}}{\tau} &= \frac{\vec{\kappa}' \times \vec{\kappa}}{\sqrt{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}}
\end{aligned}$$

$$\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) = \frac{\vec{\kappa}'' \times \vec{\kappa}}{\sqrt{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}} - \frac{1}{2}\frac{(\vec{\kappa}' \times \vec{\kappa})2\left[(\vec{\kappa}' \times \vec{\kappa}) \times (\vec{\kappa}'' \times \vec{\kappa})\right]}{\left[(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})\right]^{3/2}}$$

Collecting terms, and showing a cross product, we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) &= \frac{(\vec{\kappa}' \times \vec{\kappa}) \times [(\vec{\kappa}'' \times \vec{\kappa}) \times (\vec{\kappa}' \times \vec{\kappa})]}{[(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})]^{3/2}} \\ &= \frac{(\vec{\kappa}'' \times \vec{\kappa})}{\sqrt{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}}\end{aligned}$$

Applying our forefactor, we have

$$\begin{aligned}\left(-\frac{\vec{\tau}}{\tau}\right)\frac{d}{ds}\left(\frac{\vec{\tau}}{\tau}\right) &= -\frac{\vec{\kappa}' \times \vec{\kappa}}{\sqrt{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}}\frac{(\vec{\kappa}'' \times \vec{\kappa})}{\sqrt{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}} \\ &= \frac{(\vec{\kappa}'' \times \vec{\kappa}) \times (\vec{\kappa}' \times \vec{\kappa})}{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})} \\ &= \vec{\kappa}\frac{(\vec{\kappa}'' \cdot (\vec{\kappa}' \times \vec{\kappa})}{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}\end{aligned}$$

We now substitute in our formula for $\vec{\gamma}$,

$$\begin{aligned}\vec{\gamma} &= -\vec{\kappa} + \vec{\kappa}\left[\frac{(\vec{\kappa}'') \cdot (\vec{\kappa}' \times \vec{\kappa})}{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}\right] \\ \vec{\gamma} &= \vec{\kappa}\left[\frac{(\vec{\kappa}'' - (\vec{\kappa}' \times \vec{\kappa})) \cdot (\vec{\kappa}' \times \vec{\kappa})}{(\vec{\kappa}' \times \vec{\kappa}) \cdot (\vec{\kappa}' \times \vec{\kappa})}\right]\end{aligned}$$

The formula above explicitly shows the colinearity between $\vec{\kappa}$ and $\vec{\gamma}$.

For our final form, we substitute our expression parameterized by θ.

$$\begin{aligned}\vec{\kappa} &= \frac{\tilde{v}^*\tilde{a} - \tilde{v} \cdot \tilde{a}}{v^3} \\ \vec{\kappa}' &= \frac{d\vec{\kappa}}{ds} = \frac{1}{v}\frac{d}{d\theta}\left(\frac{\tilde{v}^*\tilde{a} - \tilde{v} \cdot \tilde{a}}{v^3}\right) \\ &= \frac{1}{v}\left[\left(\frac{\tilde{a}^*\tilde{a} + \tilde{v}^*\tilde{j} - \tilde{a} \cdot \tilde{a} - \tilde{v} \cdot \tilde{j}}{v^3}\right) + \frac{(\tilde{v}^*\tilde{a} - \tilde{v} \cdot \tilde{a})(-3)}{v^4}\frac{\tilde{v} \cdot \tilde{a}}{v}\right] \\ &= \frac{\tilde{v}^*\tilde{j} - \tilde{v} \cdot \tilde{j}}{v^4} - 3\frac{(\tilde{v}^*\tilde{a} - \tilde{v} \cdot \tilde{a})(\tilde{v} \cdot \tilde{a})}{v^6} \\ \vec{\kappa}' \times \vec{\kappa} &= \frac{(\tilde{v}^*\tilde{j} - \tilde{v} \cdot \tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v} \cdot \tilde{a})}{v^7}\end{aligned}$$

We see that the cross product eliminated parallel terms above.

We now find $\vec{\kappa}''$.

$$\begin{aligned}
\vec{\kappa}'' &= \frac{d\vec{\kappa}'}{ds} \\
&= \frac{1}{v}\frac{(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y}) + (\tilde{a}^*\tilde{j} - \tilde{a}\cdot\tilde{j})}{v^4} - \frac{1}{v}(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})(-4)v^{-5}\left(\frac{\tilde{v}\cdot\tilde{a}}{v}\right) \\
&\quad -3\frac{1}{v}\left(\frac{(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y})(\tilde{v}\cdot\tilde{a}) + (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})a^2}{v^6} - (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})(\tilde{v}\cdot\tilde{a})(-6)v^{-7}\left(\frac{\tilde{v}\cdot\tilde{a}}{v}\right)\right) \\
&= \frac{(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y}) + (\tilde{a}^*\tilde{j} - \tilde{a}\cdot\tilde{j})}{v^5} + \frac{(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})(\tilde{v}\cdot\tilde{a})}{v^5} \\
&\quad -3\frac{(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})(a^2v^2 - (\tilde{a}\cdot\tilde{v})^2)}{v^9} + 3\frac{(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})(\tilde{v}\cdot\tilde{a})^2}{v^9}
\end{aligned}$$

When we do our triple product $\vec{\kappa}''\cdot(\vec{\kappa}'\times\vec{\kappa})$, most of these terms will disappear due to orthogonality.

$$\vec{\kappa}''\cdot(\vec{\kappa}'\times\vec{\kappa}) = \frac{(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y}) + (\tilde{a}^*\tilde{j} - \tilde{a}\cdot\tilde{j})}{v^5}\cdot\frac{(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})\times(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})}{v^7}$$

We finally have our result.

$$\begin{aligned}
\vec{\gamma} &= -\vec{\kappa} + \vec{\kappa}\left[\frac{(\vec{\kappa}'')\cdot(\vec{\kappa}'\times\vec{\kappa})}{(\vec{\kappa}'\times\vec{\kappa})\cdot(\vec{\kappa}'\times\vec{\kappa})}\right] \\
&= -\vec{\kappa} + \vec{\kappa}\left[\frac{\left[(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y}) + (\tilde{a}^*\tilde{j} - \tilde{a}\cdot\tilde{j})\right]\cdot\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})\times(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]}{\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})\times(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]\cdot\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j})\times(\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]}\right]
\end{aligned}$$

Summary

Frenet Equations in Quaternion Format

$$
\begin{aligned}
\frac{d\tilde{r}}{ds} &= \tilde{u} \\
\frac{d\tilde{u}}{ds} &= \tilde{u}\vec{\kappa} \\
\frac{d\tilde{n}}{ds} &= \tilde{n}(\vec{\tau} + \vec{\kappa}) \\
\frac{d\tilde{b}}{ds} &= \tilde{b}(\vec{\gamma} + \vec{\tau}) \\
\frac{d\tilde{w}}{ds} &= \tilde{w}\vec{\gamma} \\
\tilde{n} &= \tilde{u}\left(\frac{\vec{\kappa}}{\kappa}\right) \\
\tilde{b} &= \tilde{n}\left(\frac{\vec{\tau}}{\tau}\right) \\
\tilde{w} &= \tilde{b}\left(\frac{\vec{\gamma}}{\gamma}\right) \\
\frac{d\tilde{r}}{d\theta} &= \tilde{v} \\
\frac{d\tilde{v}}{d\theta} &= \tilde{a} \\
\frac{d\tilde{a}}{d\theta} &= \tilde{j} \\
\frac{d\tilde{j}}{d\theta} &= \tilde{y} \\
ds &= v\,d\theta \\
\vec{\kappa} &= \frac{\tilde{v}^*\tilde{a} - \tilde{a}\cdot\tilde{v}}{v^3} \\
\vec{\tau} &= \frac{(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})}{v(v^2a^2 - (\tilde{v}\cdot\tilde{a})^2)} \\
\vec{\gamma} &= -\vec{\kappa} + \vec{\kappa}\left[\frac{\left[(\tilde{v}^*\tilde{y} - \tilde{v}\cdot\tilde{y}) + (\tilde{a}^*\tilde{j} - \tilde{a}\cdot\tilde{j})\right]\cdot\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]}{\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]\cdot\left[(\tilde{v}^*\tilde{j} - \tilde{v}\cdot\tilde{j}) \times (\tilde{v}^*\tilde{a} - \tilde{v}\cdot\tilde{a})\right]}\right]
\end{aligned}
$$