Quantum Mechanics, Fourier Transforms and Action

Kurt Nalty

September 2, 2008

Introduction

1 Quantum Mechanics and Action

The goal of this section is to argue that 'classical' quantum mechanics is a study of the Fourier transform of action.

1.1 Photon Energy and Spin

The energy of a photon is given by $E = \hbar\omega$, where $\hbar$ = 1.0546E-34 J s. The photon also has constant angular momentum (spin 1) of $\hbar$. Treating E as the magnitude of a phasor, we see that E looks like the Fourier transform of a derivative. Since E = da/dt defines energy in terms of action, we can claim that a photon has a constant magnitude of action, while the energy scales as the time derivative.

Comment: look at analogs between Action, Photon Energy and RLC circuits.

1.2 Quantum Operators for Energy and Momentum

In 'classical' quantum mechanics, the recipe for making the transistion from macroscopic to microscopic physics involves operators acting upon a complex valued wave function.

Energy is replaced by

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \tag{1}$$

Momentum is given by

$$\vec{p} \rightarrow \frac{\hbar}{i}\vec{\nabla} \tag{2}$$

$$p^2 \rightarrow -\hbar^2\nabla^2 \tag{3}$$

Angular momentum is given by

$$L_z \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \phi} \tag{4}$$

As seen in the definition for energy, the wavefunction as defined has units of complex action.

In practice, the wave function is usually determined from differential equations.

$$\frac{p^2}{2m} + V = E \tag{5}$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{6}$$

The quantum wave function is schematically defined as

$$\vec{\Psi}(\vec{x},t) = \int\int\int A(\vec{k})e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}dk_x dk_y dk_z \tag{7}$$

The w term is usually eliminated by a dispersion relationship, such as $w(\vec{k}) = \hbar k^2/(2m)$ for a free particle.

Compare this to the inverse Fourier transform

$$f(\vec{x},t) = \frac{1}{(2\pi)^2}\int\int\int\int F(\vec{k},\omega)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}d\omega dk_x dk_y dk_z \tag{8}$$

We see some obvious similarities and differences. The sign difference in the exponential can be dismissed as a sign convention. Likewise, the scale factor of $1/(2\pi)^2$ can also be dismissed as not important. However, the difference in integrals is very important. The classical quantum mechanics wave function is not a proper Fourier synthesis! It is instead an ad hoc synthesis. The integral over ω is missing.

What I can say with certainty about the wave function definition, is that it is the inverse three dimensional Fourier transform of the convolution of two functions.

Define

$$a(\vec{x}) = \int\int\int A(\vec{k})e^{i\vec{k}\cdot\vec{x}}dk_x dk_y dk_z \tag{9}$$

$$g(\vec{x},t) = \int\int\int e^{-i\omega(\vec{k})t}e^{i\vec{k}\cdot\vec{x}}dk_x dk_y dk_z \tag{10}$$

From the convolution relationship for Fourier transforms, we have

$$\int\int\int\left[A(\vec{k})e^{-i\omega(\vec{k})t}\right]e^{i(\vec{k}\cdot\vec{x})}dk_x dk_y dk_z = \int\int\int a(\vec{x})g(\vec{X}-\vec{x},t)dX dY dZ \tag{11}$$

This convolution integral provides a weighted measure of $a(\vec{x})$ across all space using the complex function $g(\vec{x},t)$. This convolution smears out a particle's locality, and turns a potentially real valued function complex.

In my opinion, the wavefunction is a snapshot of action at a point in time, convoluted with a planewave transform.

The wavefunction shares many useful properties of the four dimensional Fourier transform of action. The historical motivation for a wave function as opposed to the four dimensional Fourier transform may have been the elimination of the integration over time. I don't know enough of the history of QM development, and would love to learn more. At any rate, the Fourier and wave-function transforms both turn partial derivatives into complex scale factors.

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi(\vec{x},t) &= \int\int\int -i\omega A(\vec{k})e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}dk_x dk_y dk_z \\
\frac{\partial}{\partial t}f(\vec{x},t) &= \frac{1}{(2\pi)^2}\int\int\int\int -i\omega F(\vec{k},\omega)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}d\omega dk_x dk_y dk_z \\
\vec{\nabla}\Psi(\vec{x},t) &= \int\int\int -i\vec{k}A(\vec{k})e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}dk_x dk_y dk_z \\
\vec{\nabla}f(\vec{x},t) &= \frac{1}{(2\pi)^2}\int\int\int\int -i\vec{k}F(\vec{k},\omega)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}d\omega dk_x dk_y dk_z
\end{aligned}$$