Derivation of the Fibonacci Formula using the Golden Ratio

Kurt Nalty

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Abstract

A simple derivation of the Binet formula is provided.

1 Simple Derivation

The equation for the golden ratio is

$$\alpha^2 = \alpha + 1$$

which has two solutions

$$\begin{aligned} \alpha_1 &= \frac{1+\sqrt{5}}{2} = \phi \\ \alpha_2 &= \frac{1-\sqrt{5}}{2} = 1-\phi \end{aligned}$$

Raising α to integral powers, and substituting for lower powers yields

$$\begin{aligned} \alpha^2 &= \alpha + 1 \\ \alpha^3 &= \alpha^2 + \alpha = 2\alpha + 1 \\ \alpha^4 &= \alpha^3 + \alpha^2 = 3\alpha + 2 \\ \alpha^5 &= \alpha^4 + \alpha^3 = 5\alpha + 3 \\ \alpha^6 &= \alpha^5 + \alpha^4 = 8\alpha + 5 \\ \alpha^n &= F(n)\alpha + F(n-1) \end{aligned}$$

Using our two solutions for α, we have

$$\begin{aligned} \alpha^n &= F(n)\alpha + F(n-1) \\ \phi^n &= F(n)\phi + F(n-1) \\ (1-\phi)^n &= F(n)\,(1-\phi) + F(n-1) \\ \phi^n - (1-\phi)^n &= F(n)\phi - F(n)\,(1-\phi) \\ F(n) &= \frac{\phi^n - (1-\phi)^n}{\phi - (1-\phi)} \end{aligned}$$

$$
\begin{aligned}
F(n) &= \frac{\phi^n - (1-\phi)^n}{2\phi - 1} \\
F(n) &= \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}
\end{aligned}
$$