Extending the Euler Equation Beyond Velocity

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Introduction

The traditional Euler equation deals with potential which are functions only of position and velocity. While fine for mechanical systems, this is inadequate for electromagnetic systems where radiation reaction requires acceleration and jerk terms to be included.

Traditional Euler Equation

The traditional Euler equation is the foundation of classical mechanics.

A Derivation of the Usual Euler Equation

Euler's equation is usually derived using only position and velocity dependent terms.

Begin with an energy function.

$$E = \int_{t_1}^{t_2} P\left(x, v, t\right) dt$$

The variation of energy balance versus particle path is assumed minimized when following the actual path. We present candidate paths $X(t)$ which differ from the actual path $x(t)$ by an arbitrary function of time $\alpha\varphi\left(t\right)$, chosen to be zero at the start and stop times, parameterized by a scale factor α, which is not a function of time.

$$X\left(t\right) = x\left(t\right) + \alpha\varphi\left(t\right)$$

$$\varphi\left(t_1\right) = \varphi\left(t_2\right) = 0$$

Various derivatives of use are

$$\dot{X}(t) = \dot{x}(t) + \alpha\dot{\varphi}(t)$$
$$\frac{dX}{d\alpha} = \frac{\partial X}{\partial \alpha} = \varphi(t)$$
$$\frac{d\dot{X}}{d\alpha} = \frac{\partial \dot{X}}{\partial \alpha} = \dot{\varphi}(t)$$
$$\frac{dt}{d\alpha} = \frac{dx}{d\alpha} = \frac{dv}{d\alpha} = 0$$

Minimizing E with respect to α requires $dE/d\alpha = 0$.

$$\begin{aligned}
\frac{dE}{d\alpha} &= \int_{t_1}^{t_2} \left(\frac{dP}{d\alpha}\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial P}{\partial X}\frac{dX}{d\alpha} + \frac{\partial P}{\partial \dot{X}}\frac{d\dot{X}}{d\alpha} + \frac{\partial P}{\partial t}\frac{dt}{d\alpha}\right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial P}{\partial X}\varphi + \frac{\partial P}{\partial \dot{X}}\dot{\varphi} + 0\right) dt \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \int_{t_1}^{t_2} \frac{\partial P}{\partial \dot{X}}\dot{\varphi} dt \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \int_{t_1}^{t_2} \frac{\partial P}{\partial \dot{X}} d\varphi \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \left[\frac{\partial P}{\partial \dot{X}}\varphi\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \varphi d\left(\frac{\partial P}{\partial \dot{X}}\right) \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt - \int_{t_1}^{t_2} \varphi \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right) dt \\
&= \int_{t_1}^{t_2} \varphi \left(\frac{\partial P}{\partial X} - \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right)\right) dt
\end{aligned}$$

The integration by parts term

$$\left[\frac{\partial P}{\partial \dot{X}}\varphi\right]_{t_1}^{t_2}$$

disappears as ϕ is chosen to be zero at the endpoints. For the real path, $x = X$ and $v = \dot{X}$. As ϕ is arbitrary, this requires

$$\frac{\partial P}{\partial x} - \frac{d}{dt}\left(\frac{\partial P}{\partial v}\right) = 0$$

or equivalently

$$\frac{\partial P}{\partial x} = \frac{d}{dt}\left(\frac{\partial P}{\partial v}\right)$$

In this discussion, I've used energy, power, and spatial coordinates and velocities simply because I like these terms due to my engineering background. The mathematics however, are not restricted to these particular terms. These apply to any function extremalized, that can be expressed in terms of coordinates and first derivatives of coordinates. As an example, the E above could be replaced by an action term, or the P could be replaced by an energy function, or t could be replaced by pathlength s, and the same relationships would hold.

Euler Equation Examples

Ballistics

As an example of the Euler equation in mechanics, look at the equations of motion for a conservative system consisting of a mass M moving at velocity v in a constant gravitational field.

$$E = \frac{1}{2}mv^2 - mgz = m\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - gz\right)$$

Energy is conserved and constant in this example.

$$\begin{aligned}
\frac{\partial E}{\partial x} &= \frac{\partial E}{\partial y} = 0 \\
\frac{\partial E}{\partial z} &= -mg \\
\frac{\partial E}{\partial v_x} &= \frac{\partial E}{\partial v_x} = 0 \\
\frac{\partial E}{\partial v_z} &= \frac{1}{2}mv_z \\
\frac{d}{dt}\left(\frac{\partial E}{\partial v_x}\right) &= \frac{d}{dt}\left(\frac{\partial E}{\partial v_y}\right) = 0 \\
\frac{d}{dt}\left(\frac{\partial E}{\partial v_z}\right) &= \frac{1}{2}ma_z = \frac{\partial E}{\partial z} = -mg
\end{aligned}$$

Electromagnetics

A charged particle with mass m and charge q in a field with potentials ϕ and $\vec{A}$ has an energy function U given by

$$U = \frac{1}{2}mv^2 - q\left(\phi - \vec{v}\cdot\vec{A}\right)$$

We calculate the spatial partial derivatives

$$\begin{aligned}
\nabla U &= -q\nabla\phi + q\nabla\left(\vec{v}\cdot\vec{A}\right) \\
&= q\left(-\nabla\phi + \vec{v}\times\left(\nabla\times\vec{A}\right) + \left(\vec{v}\cdot\nabla\right)\vec{A}\right) \\
&= q\left(-\nabla\phi + \vec{v}\times\left(\nabla\times\vec{A}\right) + \frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t}\right) \\
&= q\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times\left(\nabla\times\vec{A}\right) + \frac{d\vec{A}}{dt}\right) \\
&= q\left(\vec{E} + \vec{v}\times\vec{B} + \frac{d\vec{A}}{dt}\right)
\end{aligned}$$

and the velocity partial derivatives

$$\begin{aligned}
\nabla_v U &= m\vec{v} + q\vec{A} \\
\frac{d}{dt}\left(\nabla_v U\right) &= m\vec{a} + q\frac{d\vec{A}}{dt}
\end{aligned}$$

Equating ∇U with $d\left(\nabla_v U\right)/dt$ yields

$$\begin{aligned}
q\left(\vec{E} + \vec{v}\times\vec{B} + \frac{d\vec{A}}{dt}\right) &= m\vec{a} + q\frac{d\vec{A}}{dt} \\
q\left(\vec{E} + \vec{v}\times\vec{B}\right) &= m\vec{a}
\end{aligned}$$

So far, so good. However, once we begin to look at radiative processes, we find difficulty with this classical approach to electromagnetics.

Radiation Reaction

Using the Maxwell equations and solving for far field radiated power, we find that the power radiated by an accelerated charge is proportional to the acceleration squared. (Larmor formula)

$$P = \frac{2}{3}q^2a^2$$

Using assumptions that this is a time averaged power covering one cycle of oscillation, Abrahams and Lorentz developed a radiation reaction formula

$$m\vec{a} = \frac{2}{3}q^2\frac{d\vec{a}}{dt}$$

This formula has run away (tachyon) solutions. This formula also is non-covariant and thus relativistically impossible.

Rohrlich develops a covariant formula for radiation reaction. In the Lorentz-Dirac equation below, the dots indicate derivatives with respect to proper time.

$$ma^{\mu} = qF^{\mu\nu}v_{\nu} + \frac{2}{3}q^{2}\left(\dot{a}^{\mu} - a^{\lambda}a_{\lambda}v^{\mu}\right)$$

While dealing with the covariant objections, this equation still has difficulties with run-away solutions, and poses problems when trying to obtain the force equations from a dynamical principle.

Radiation reaction indicates that we must develop equations of motion based upon at least $d\vec{a}/dt$ (jerk), and that we need to expand the Euler equation to deal with acclerations in the energy terms.

Extending the Euler Equation

Following the lead of Arfken (Mathematical Methods for Physicists), we extend the Euler equations to include acceleration and higher order terms required for electromagnetics

Euler's equation can be expanded to include acceleration dependent terms. The reference for this formula is Arfkin, Mathematical Methods for Physicist, Third edition, p. 929.

Begin with an energy function.

$$E = \int_{t_1}^{t_2} P\left(x, v, a, t\right) dt$$

The variation of energy balance versus particle path is assumed minimized when following the actual path. We present candidate paths which differ from the actual path by an arbitrary function of time, chosen to be zero at the start and stop times, parameterized by α, which is not a function of time.

$$X\left(t\right) = x\left(t\right) + \alpha\varphi\left(t\right)$$

$$\varphi\left(t_1\right) = \varphi\left(t_2\right) = 0$$

$$\varphi'\left(t_1\right) = \varphi'\left(t_2\right) = 0$$

Various derivatives of use are

$$
\begin{aligned}
\dot{X}(t) &= \dot{x}(t) + \alpha\dot{\varphi}(t) \\
\ddot{X}(t) &= \ddot{x}(t) + \alpha\ddot{\varphi}(t) \\
\frac{dX}{d\alpha} &= \frac{\partial X}{\partial\alpha} = \varphi(t) \\
\frac{d\dot{X}}{d\alpha} &= \frac{\partial\dot{X}}{\partial\alpha} = \dot{\varphi}(t) \\
\frac{d\ddot{X}}{d\alpha} &= \frac{\partial\ddot{X}}{\partial\alpha} = \ddot{\varphi}(t) \\
\frac{dt}{d\alpha} = \frac{dx}{d\alpha} = \frac{dv}{d\alpha} = \frac{da}{d\alpha} &= 0
\end{aligned}
$$

Minimizing E with respect to α requires $dE/d\alpha = 0$.

$$
\begin{aligned}
\frac{dE}{d\alpha} &= \int_{t_1}^{t_2} \left(\frac{dP}{d\alpha}\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial P}{\partial X}\frac{dX}{d\alpha} + \frac{\partial P}{\partial \dot{X}}\frac{d\dot{X}}{d\alpha} + \frac{\partial P}{\partial \ddot{X}}\frac{d\ddot{X}}{d\alpha} + \frac{\partial P}{\partial t}\frac{dt}{d\alpha}\right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial P}{\partial X}\varphi + \frac{\partial P}{\partial \dot{X}}\dot{\varphi} + \frac{\partial P}{\partial \ddot{X}}\ddot{\varphi} + 0\right) dt \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \int_{t_1}^{t_2} \frac{\partial P}{\partial \dot{X}}\dot{\varphi} dt + \int_{t_1}^{t_2} \frac{\partial P}{\partial \ddot{X}}\ddot{\varphi} dt \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \int_{t_1}^{t_2} \frac{\partial P}{\partial \dot{X}} d\varphi + \int_{t_1}^{t_2} \frac{\partial P}{\partial \ddot{X}} d\dot{\varphi} \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt + \left[\frac{\partial P}{\partial \dot{X}}\varphi\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \varphi d\left(\frac{\partial P}{\partial \dot{X}}\right) + \left[\frac{\partial P}{\partial \ddot{X}}\dot{\varphi}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{\varphi} d\left(\frac{\partial P}{\partial \ddot{X}}\right) \\
&= \int_{t_1}^{t_2} \frac{\partial P}{\partial X}\varphi dt - \int_{t_1}^{t_2} \varphi \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right) dt - \int_{t_1}^{t_2} \frac{d\varphi}{dt} d\left(\frac{\partial P}{\partial \ddot{X}}\right) \\
&= \int_{t_1}^{t_2} \varphi \left(\frac{\partial P}{\partial X} - \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right)\right) dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial P}{\partial \ddot{X}}\right) d\varphi \\
&= \int_{t_1}^{t_2} \varphi \left(\frac{\partial P}{\partial X} - \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right)\right) dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial P}{\partial \ddot{X}}\right) d\varphi \\
&= \int_{t_1}^{t_2} \varphi \left(\frac{\partial P}{\partial X} - \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right)\right) dt - \left[\frac{d}{dt}\left(\frac{\partial P}{\partial \ddot{X}}\right)\varphi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \varphi \frac{d^2}{dt^2}\left(\frac{\partial P}{\partial \ddot{X}}\right) dt \\
&= \int_{t_1}^{t_2} \varphi \left(\frac{\partial P}{\partial X} - \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{X}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial P}{\partial \ddot{X}}\right)\right) dt
\end{aligned}
$$

The integration by parts term

$$\left[\frac{\partial P}{\partial \dot{X}}\varphi\right]_{t_1}^{t_2}$$

disappears as ϕ is chosen to be zero at the endpoints. Likewise, the term

$$\left[\frac{\partial P}{\partial \ddot{X}}\dot{\varphi}\right]_{t_1}^{t_2}$$

disappears as $\dot{\varphi}$ is chosen to be zero at the endpoints. For the real path, $x = X$, $v = \dot{X}$, and $a = \ddot{X}$. As ϕ is arbitrary, this requires

$$\frac{\partial P}{\partial x} - \frac{d}{dt}\left(\frac{\partial P}{\partial v}\right) + \frac{d^2}{dt^2}\left(\frac{\partial P}{\partial a}\right) = 0$$

or equivalently

$$\frac{\partial P}{\partial x} - \frac{d}{dt}\left(\frac{\partial P}{\partial v}\right) = -\frac{d^2}{dt^2}\left(\frac{\partial P}{\partial a}\right)$$

In a similar fashion, higher derivates can be used in energy, and similar terms of alternating sign appendend.

For example, for $E(x, v, a, j, y; t)$, where j is jerk and y is yank, we have

$$\frac{\partial P}{\partial x} - \frac{d}{dt}\left(\frac{\partial P}{\partial v}\right) + \frac{d^2}{dt^2}\left(\frac{\partial P}{\partial a}\right) - \frac{d^3}{dt^3}\left(\frac{\partial P}{\partial j}\right) + \frac{d^4}{dt^4}\left(\frac{\partial P}{\partial y}\right) = 0$$

Noncommutative Derivatives

When working with partial and full derivatives, we must keep an awareness of the non-commutative nature of full and partial derivatives.

Begin by examining a candidate function $f = f(x, v, a, t)$. The full derivative of this function is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial t}dt$$

Dividing by dt, this leads to

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}v + \frac{\partial f}{\partial v}a + \frac{\partial f}{\partial a}j + \frac{\partial f}{\partial t} \\ \frac{df}{dt} &= v\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial v} + j\frac{\partial f}{\partial a} + \frac{\partial f}{\partial t} \end{aligned}$$

Operationally, for this particular functional form for f, we could say

$$\frac{d}{dt} = \left(v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + j\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\right)$$

Now examine the various partial derivatives of the time derivative of f.

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{df}{dt}\right) &= \frac{\partial}{\partial x}\left(v\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial x}\left(a\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial x}\left(j\frac{\partial f}{\partial a}\right) + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) \\ &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) \end{aligned}$$

However,

$$\begin{aligned} \frac{\partial}{\partial v}\left(\frac{df}{dt}\right) &= \frac{\partial}{\partial v}\left(v\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial v}\left(a\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial v}\left(j\frac{\partial f}{\partial a}\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial t}\right) \\ &= \left(\frac{\partial f}{\partial x}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) \end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial a}\left(\frac{df}{dt}\right) &= \frac{\partial}{\partial a}\left(v\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial a}\left(a\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial a}\left(j\frac{\partial f}{\partial a}\right) + \frac{\partial}{\partial a}\left(\frac{\partial f}{\partial t}\right) \\
&= \left(\frac{\partial f}{\partial v}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial a}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial j}\left(\frac{df}{dt}\right) &= \frac{\partial}{\partial j}\left(v\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial j}\left(a\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial j}\left(j\frac{\partial f}{\partial a}\right) + \frac{\partial}{\partial j}\left(\frac{\partial f}{\partial t}\right) \\
&= \left(\frac{\partial f}{\partial a}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial j}\right)
\end{aligned}
$$

This leads to the commutator relationships

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{df}{dt}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) &= 0 \\
\frac{\partial}{\partial v}\left(\frac{df}{dt}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial v}\right) &= \left(\frac{\partial f}{\partial x}\right) \\
\frac{\partial}{\partial a}\left(\frac{df}{dt}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial a}\right) &= \left(\frac{\partial f}{\partial v}\right) \\
\frac{\partial}{\partial j}\left(\frac{df}{dt}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial j}\right) &= \left(\frac{\partial f}{\partial a}\right)
\end{aligned}
$$

For this example, where $f = f(x, v, a; t)$, this last equation simplifies to

$$
\frac{\partial}{\partial j}\left(\frac{df}{dt}\right) = \left(\frac{\partial f}{\partial a}\right)
$$

The bottom line is that full and partial derivatives for functions including velocity, acceleration, and higher derivatives do not commute.

Action, Energy and Euler Equations

Action is far more important as a fundamental of physics than is currently appreciated. The Euler equations before have been written using energy. A useful exercise is to express Energy as the time derivative of Action, and see the new form of the Euler equation.

For this example, begin with the extended Euler equation for $E(a, v, x, t)$.

$$
\frac{\partial E}{\partial x} - \frac{d}{dt}\left(\frac{\partial E}{\partial v}\right) + \frac{d^2}{dt^2}\left(\frac{\partial E}{\partial a}\right) = 0
$$

Substitute $E = dA/dt$. (The A stands for action.)

$$\frac{\partial}{\partial x}\left(\frac{dA}{dt}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v}\frac{dA}{dt}\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\frac{dA}{dt}\right) = 0$$

Using the results of the previous section, we can substantially simplify this expression.

The first term above commutes, leading to

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v}\frac{dA}{dt}\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\frac{dA}{dt}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x} - \left(\frac{\partial}{\partial v}\frac{dA}{dt}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial a}\frac{dA}{dt}\right)\right) = 0$$

The middle term expands to

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial x} - \frac{d}{dt}\left(\frac{\partial A}{\partial v}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial a}\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^2}{dt^2}\left(-\frac{\partial A}{\partial v} + \frac{\partial}{\partial a}\frac{dA}{dt}\right) = 0$$

The right hand term expands,

$$\frac{d^2}{dt^2}\left(-\frac{\partial A}{\partial v} + \frac{\partial A}{\partial v} + \frac{d}{dt}\frac{\partial A}{\partial a}\right) = 0$$

$$\frac{d^3}{dt^3}\left(\frac{\partial A}{\partial a}\right) = 0$$

This is very easy to solve. Expressing energy functions in terms of Action derivatives leads to a very simple form for Euler's equations. Now, let's look at a more complicated form for potential.

Repeating this process for $E(x, v, a, j, y; t)$, we have

$$\frac{\partial E}{\partial x} - \frac{d}{dt}\left(\frac{\partial E}{\partial v}\right) + \frac{d^2}{dt^2}\left(\frac{\partial E}{\partial a}\right) - \frac{d^3}{dt^3}\left(\frac{\partial E}{\partial j}\right) + \frac{d^4}{dt^4}\left(\frac{\partial E}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{dA}{dt}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v}\left(\frac{dA}{dt}\right)\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\left(\frac{dA}{dt}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v}\left(\frac{dA}{dt}\right)\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\left(\frac{dA}{dt}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x} - \frac{\partial}{\partial v}\left(\frac{dA}{dt}\right)\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\left(\frac{dA}{dt}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial x} - \frac{d}{dt}\left(\frac{\partial A}{\partial v}\right)\right) + \frac{d^2}{dt^2}\left(\frac{\partial}{\partial a}\left(\frac{dA}{dt}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^2}{dt^2}\left(-\frac{\partial A}{\partial v} + \frac{\partial}{\partial a}\left(\frac{dA}{dt}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^2}{dt^2}\left(-\frac{\partial A}{\partial v} + \frac{\partial A}{\partial v} + \frac{d}{dt}\left(\frac{\partial A}{\partial a}\right)\right) - \frac{d^3}{dt^3}\left(\frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^3}{dt^3}\left(\frac{\partial A}{\partial a} - \frac{\partial}{\partial j}\left(\frac{dA}{dt}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^3}{dt^3}\left(\frac{\partial A}{\partial a} - \frac{\partial A}{\partial a} - \frac{d}{dt}\left(\frac{\partial A}{\partial j}\right)\right) + \frac{d^4}{dt^4}\left(\frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^4}{dt^4}\left(-\frac{\partial A}{\partial j} + \frac{\partial}{\partial y}\left(\frac{dA}{dt}\right)\right) = 0$$

$$\frac{d^4}{dt^4}\left(-\frac{\partial A}{\partial j} + \frac{\partial A}{\partial j} + \frac{d}{dt}\left(\frac{\partial A}{\partial y}\right)\right) = 0$$

$$\frac{d^5}{dt^5}\left(\frac{\partial A}{\partial y}\right) = 0$$

Once again, this is very easy to solve. Expressing energy functions in terms of Action derivatives leads to a very simple form for Euler's equations. Future work is to take this expression, and instead of parameterization by time, use four-vector distance s as the parameter, and examine the resulting equations from the point of view of curvatures, torsion and lift parameters.

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